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Partial $\mathcal{N}=2 \to \mathcal{N}=1$ supersymmetry breaking and gravity deformed chiral rings

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Abstract: We present a derivation of the chiral ring relations, arising in $\mathcal{N}=1$ gauge theories in the presence of (anti-)self-dual background gravitational field $G_{\alpha\beta\gamma}$ and graviphoton field strength $F_{\alpha\beta}$. These were previously considered in the literature in order to prove the relation between gravitational F-terms in the gauge theory and coefficients of the topological expansion of the related matrix integral. We consider the spontaneous breaking of $\mathcal{N}=2$ to $\mathcal{N}=1$ supergravity coupled to vector- and hyper-multiplets, and take a rigid limit which keeps a non-trivial $G_{\alpha\beta\gamma}$ and $F_{\alpha\beta}$ with a finite supersymmetry breaking scale. We derive the resulting effective, global, $\mathcal{N}=1$ theory and show that the chiral ring relations are just a consequence of the standard $\mathcal{N}=2$ supergravity Bianchi identities. We can also obtain models with matter in different representations and in particular quiver theories. We also show that, in the presence of non-trivial $F_{\alpha\beta}$, consistency of the Konishi-anomaly loop equations with the chiral ring relations, demands that the gauge kinetic function and the superpotential, a priori unrelated for an $\mathcal{N}=1$ theory, should be derived from a prepotential, indicating an underlying $\mathcal{N}=2$ structure.

Keywords: M(atrix) Theories, Supersymmetry Breaking, Extended Supersymmetry, Supergravity Models.

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Contents

1. Introduction

In the last year there has been a considerable progress in computing glueball superpotentials in $\mathcal{N} = 1$ Super Yang-Mills (SYM) theories, as a result of the discovery made in [1] that such computations can be efficiently organized by a matrix model integral. Whereas the leading planar term of the latter is related to ordinary F-terms in the gauge theory, higher genus contributions in the topological expansion of the matrix model capture certain generalized gravitational F-terms, originally studied in the context of type-II string theories on Calabi-Yau manifolds [2, 3]. In particular, the genus 1 term computes an F-term involving the gravitational background $G_{\alpha\beta\gamma}$ to which the gauge theory couples. The non-trivial gravitational background is responsible for a deformation of the basic chiral ring relation among the gauginos of the $\mathcal{N} = 1$ SYM theory, as can be seen by starting from the Bianchi identities of $\mathcal{N} = 1$ supergravity coupled to super Yang-Mills. This fact plays a crucial role in the proof of the identification of the superpotential term proportional to G^2 with the leading non-planar term in the matrix model integral.

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As for the $g > 1$ terms however, the generalized superpotentials necessarily involve the graviphoton field strength $F_{\alpha\beta}$ and therefore cannot be understood within the framework of $\mathcal{N} = 1$ supergravity and the related tensor calculus. In [4, 5] it has been proposed, with string theoretic motivations, that the basic chiral ring relation, stating the Grassmann nature of the gluino superfield W_α, is deformed by the presence of a non-trivial graviphoton field strength, whereas superspace coordinates are unaffected. A different viewpoint has been taken in [6, 7], where instead the Grassmann nature of the superspace coordinates is deformed.

The aim of this paper is to reconsider the above issue. In [8] we have shown that the chiral ring relations one obtains for gauge invariant fields from the deformed chiral ring relation $\{W_\alpha, W_\beta\} = F_{\alpha\beta} + 2G_{\alpha\beta\gamma}W^\gamma$, can be obtained from the Bianchi identities of $\mathcal{N} = 2$ supergravity coupled to abelian vector multiplets and spontaneously broken by fluxes, in accordance with the open string (gauge theory)/closed string duality.

We would like to follow the same strategy on the super Yang-Mills side (we will discuss also the possibility that this picture may arise from wrapped D5-branes effective action). The idea here is to start from $\mathcal{N} = 2$ supergravity coupled to vector multiplets in the adjoint representation of a gauge group, that will be taken to be $\mathrm{U}(N)$ (we will consider afterwards also hypermultiplets in the fundamental of $\mathrm{U}(N)$), break it spontaneously to $\mathcal{N} = 1$, following the procedure in [9, 10] of gauging two commuting translation isometries of the hypermultiplet quaternionic manifold $\mathrm{SO}(4,1)/\,\mathrm{SO}(4)$, and then take a scaling limit that decouples the quantum gravitational sector and hypermultiplet sector, while keeping a non trivial background for $F_{\alpha\beta}$ and $G_{\alpha\beta\gamma}$. This involves taking the Planck mass m_P to infinity, with finite supersymmetry braking scale Λ and scaling consistently the various fields. The resulting theory is an effective $\mathcal{N} = 1$ $\mathrm{U}(N)$ gauge theory with a non trivial superpotential for the adjoint chiral superfield Φ, coupled to the gravitational background of F and G. Then, starting from the Bianchi identities of $\mathcal{N} = 2$ supergravity [11] and expanding around the chosen vacuum we will derive precisely the chiral ring relation mentioned in the previous paragraph. However, before claiming that we have obtained the super Yang-Mills theory considered in [1], we have to settle two issues: the first is that, since our $\mathcal{N} = 1$ SYM theory comes from a spontaneously broken $\mathcal{N} = 2$ theory, a non-trivial superpotential for Φ implies also a non-trivial, Φ-dependent gauge function, which is related to the superpotential in a specific way, since both are obtained from a prepotential. The second problem is that the $\mathcal{N} = 2$ Bianchi identities that give the wanted chiral ring relation, also give a relation of the form $[W_\alpha, \Phi] \sim F_{\alpha\beta}W^\beta$, i.e. W and Φ do not commute in the ring.

Quite remarkably, the two problems cure each other: Φ can be redefined by an appropriate shift proportional to W^2 in such a way that it commutes with W. Then, after performing the same shift in the effective action, i.e. in the superpotential and in the gauge function, one can show that all the non-trivial Φ and Λ dependence of the bare gauge function cancels in the chiral ring, therefore leaving us with the SYM theory considered in [1] along with a gravity induced gauge function. The gravity induced gauge function is proportional the gravitational field strengths and has a nontrivial Φ and Λ dependence.

We should stress here an important result of our analysis: in presence of a non-trivial $F_{\alpha\beta}$, consistency of the loop equations with the chiral ring relations, demands that the gauge function, a priori arbitrary for an $\mathcal{N} = 1$ super Yang-Mills theory, must be the one given by $\mathcal{N} = 2$ supersymmetry. We regard this fact as a strong indication that our interpretation in terms of spontaneously broken $\mathcal{N} = 2$ supergravity is the correct one.

Finally, of course, the underlying $\mathcal{N} = 2$ structure also implies that the adjoint scalar superfield comes with a non-trivial kinetic Kähler potential, which gives rise to additional non-renormalizable interactions. However, from the point of view of $\mathcal{N} = 1$ super Yang-Mills, this is a D-term and therefore is not expected to modify the loop equations in the chiral ring coming from the generalized Konishi anomaly.

The above strategy can be extended to include the case of matter in the (anti-)fundamental representation of the gauge group [12]. To cover this case one has to add to the previous case hypermultiplets. We start with hypermultiplets parametrizing the quaternionic manifold $\mathrm{U}(2, N+1)/\,\mathrm{U}(2) \times \mathrm{U}(N+1)$ and vector multiplets in the $\mathrm{U}(1) \times \mathrm{U}(N)$ and gauge appropriately the hypermultiplets [13] to achieve supersymmetry breaking and $\mathrm{U}(N)$-charged fundamental hypermultiplets. In this way one gets, after decoupling, an effective $\mathcal{N} = 1$ super Yang-Mills theory with a trilinear coupling of Φ to chiral superfields in the $\mathbf{N}$ and $\bar{\mathbf{N}}$. Again, considering the Bianchi identities involving hypermultiplets, one can show that the chiral ring relations involving fundamentals are unmodified by the non trivial gravitational background. Other gauge groups and matter representations can also be incorporated in the discussion. In particular, quiver theories can also be obtained in this way.

An obvious question finally arises, as to whether the above picture can be given a D-brane interpretation. In the closed/open string duality of [14], we are dealing, in the open string side, with D5-branes wrapped on two-cycles inside certain non-compact Calabi-Yau manifolds. We have seen, in the field theoretic discussion, that what triggers partial supersymmetry breaking is the gauging of two translation isometries in the hypermultiplet's manifold. That is, more precisely, certain scalars in hypermultiplets are charged under the graviphoton U(1) and the "center of mass" U(1) in $\mathrm{U}(N)$. Whereas the former is a bulk field, it is natural to identify the latter with the center of mass degree of freedom of the D5-branes. In fact one can show that there is the correct coupling of the center of mass U(1) with hypermultiplets from the closed string sector (this was also noticed in [15]). However, due to the non-compactness of the Calabi-Yau manifold, the issue of whether there is a gauging by the graviphoton is not clear.

The paper is organized as follows: in section 2 we review the gauging formalism in $\mathcal{N} = 2$ supergravity coupled to vector- and hyper-multiplets and discuss the gauging, which gives rise to partial supersymmetry breaking, in the case where only $\mathrm{U}(N)$ vector multiplets are charged, after taking the rigid limit $m_P \to \infty$. In section 3 we also explicitly write down the resulting $\mathcal{N} = 1$ effective action. In section 4 we include hypermultiplets valued in the $\mathrm{U}(2, N+1)/\,\mathrm{U}(2) \times \mathrm{U}(N+1)$ quaternionic manifold, discuss their gauging, and the resulting supersymmetry breaking pattern. We show that in the rigid limit we obtain the coupling of $\mathbf{N}$ and $\bar{\mathbf{N}}$ chiral multiplets to the adjoint multiplet Φ and to the gauge multiplet W. In section 5 we derive the chiral ring relations from the solution of the Bianchi identities of $\mathcal{N} = 2$ supergravity of [11] after expanding around the vacuum which partially breaks supersymmetry. In section 6 we discuss the generalized Konishi anomaly equations and show that, in the presence of $F_{\alpha\beta}$, consistency demands that the gauge function and the superpotential should both be derived from a prepotential, indicating an underlying $\mathcal{N}=2$ structure.

In appendix A we explain the conventions and notation we follow for the hypermultiplets, while in appendix B we prove that after the field redefinition for Φ the gauge function become trivial in the chiral ring. Finally in appendix C we evaluate the corrections to the Konishi anomaly equations due to the presence of the gravity induced gauge kinetic term.

2. $\mathcal{N}=2$ gravity, gaugings and the rigid limit

In this section we provide the details of the $\mathcal{N}=2$ supergravity coupled with U(N) vector multiplets, the gaugings of the vector multiplet manifold and the scalings of the fields with the Planck mass to obtain $\mathcal{N}=1$ theory in the rigid limit. The details of the $\mathcal{N}=2$ supergravity action along with the gaugings of the matter fields were developed in a series of works by various authors [16]–[20], for the purposes of this paper we use the lagrangian and the conventions of [11]. Partial breaking of $\mathcal{N}=2$ to $\mathcal{N}=1$ requires gauging a minimal hypermultiplet manifold by the U(1) corresponding to the graviphoton and the overall U(1) [9]. For concreteness we take the minimal hypermultiplet manifold to be SO(4,1)/ SO(4). The theory we discuss below is the generalization of the one studied by [10] for the case of a U(1) vector multiplet. We obtain the rigid limit by scaling the Planck mass to infinity in such a way so as to retain a non dynamical supergravity background. This method provides a natural way to derive the couplings of the background $\mathcal{N}=2$ supergravity fields to the matter of the $\mathcal{N}=1$ theory after spontaneous breaking of supersymmetry.

2.1 Field content

$\mathcal{N}=2$ supergravity consists of the gravity multiplet and the matter multiplets. The gravity multiplet contains the graviton, two gravitinos ψ^A_μ, with $A=1,2$ and a U(1) gauge field called the graviphoton. The matter multiplets are of two types, the vector multiplets and the hypermultiplets. We provide the details of this sector below.

Vector multiplets. The scalars of the U(N) vector multiplets parametrize a $2N^2$ dimensional special Kähler manifold which is specified by following holomorphic section.

$$\begin{aligned}
X^0(z) &= \frac{1}{\sqrt{2}}, & F_0(z) &= -\frac{i}{\sqrt{2}}\left(2f(z) - z^a\frac{\partial f(z)}{\partial z^a}\right),\\
X^i(z) &= \frac{z^i}{\sqrt{2}} & F_i(z) &= -\frac{i}{\sqrt{2}}\frac{\partial f(z)}{\partial z^i},\\
X^n(z) &= \frac{i}{\sqrt{2}}\frac{\partial f(z)}{\partial z^n}, & F_n(z) &= \frac{z^n}{\sqrt{2}}.
\end{aligned} \tag{2.1}$$

Here the z's are complex coordinates with indices i,j which take values from $1,\dots,(n-1)$, while a,b take values from $1,\dots,n$ and $n=N^2$, the number of generators of U(N). The i,j indices parametrize the SU(N) directions while the n index refers to the overall U(1) of U(N). We normalize the Killing metric of SU(N) such that $\mathrm{Tr}(T_iT_j)=\delta_{ij}$, where T_i

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stands for $N \times N$ hermitean matrices which generate $\mathrm{SU}(N)$. The last holomorphic section in (2.1) is necessary to make the structure group of the $\mathcal{N}=2$ theory $\mathrm{U}(N)$. The action of the group $\mathrm{U}(N)$ on the coordinates z^a is given by

$$UzU^\dagger\,, \qquad \text{where } z = (z^i T_i + z^n \frac{I}{\sqrt{N}})\,, \tag{2.2}$$

where I is the identity matrix. $f(z)$ is any function which is invariant under the action of $\mathrm{U}(N)$ on the coordinates z. $\mathcal{N}=2$ supersymmetry restricts the scalar manifold to be special Kähler, therefore these sections must satisfy

$$X^\Lambda \partial_a F_\Lambda - F_\Lambda \partial_a X^\Lambda = 0 \tag{2.3}$$

here $\Lambda = 0, \ldots, n$. It is easy to verify from the ansätz for the holomorphic symplectic sections given in (2.1), the above condition is obeyed. The Kähler potential is given by

$$\begin{aligned} K &= -\log i(X^{*\Lambda} F_\Lambda - X^\Lambda F^*_\Lambda)\,, \\ &= -\log\left[(f + f^*) - \frac{1}{2}(z^a - z^{a*})(\partial_a f - (\partial_a f)^*)\right], \end{aligned} \tag{2.4}$$

from which the the metric on the manifold is constructed by $g_{ab*} = \partial_a \partial_{b*} K$.

We now discuss the gauging of the vector multiplet manifold. To perform the gauging we need to construct the Killing vectors of the manifold. From the construction of the symplectic sections in (2.1) and the adjoint action of $\mathrm{U}(N)$ on the coordinates z^a given in (2.2), we see that the following are the N^2-1 killing vectors corresponding to the $\mathrm{SU}(N)$ subgroup of $\mathrm{U}(N)$.

$$\begin{aligned} k_i^k \partial_k &= f_{ij}^k z^j \partial_k\,, \qquad k_i^{k*} \partial_{k*} = f_{ij}^k z^{j*} \partial_{k*}\,, \\ \mathbf{k}_i &= k_i^k \partial_k + k_i^{k*} \partial_{k*}\,. \end{aligned} \tag{2.5}$$

Here f_{ab}^c are the structure constants satisfying

$$[T_i, T_j] = i f_{ij}^k T_k\,, \tag{2.6}$$

note that the killing vector satisfies the following commutation relation

$$[\mathbf{k}_i, \mathbf{k}_j] = -f_{ij}^k \mathbf{k}_k\,. \tag{2.7}$$

$\mathcal{N}=2$ supersymmetry requires that the symplectic sections transform in the adjoint representation of $\mathrm{U}(N)$, this is seen from the following equations

$$k_i^k \partial_k X^j = f_{ik}^j X^k\,, \qquad k_i^k \partial_k F_j = -f_{ij}^k F_k\,. \tag{2.8}$$

These equation ensures that the following constraints are satisfied

$$k_i^j X^i = k_{i*}^{j*} X^{*i} = 0\,. \tag{2.9}$$

The prepotential functions which generate these Killing vectors

$$\mathcal{P}_i = -i k_i^j \partial_j K\,. \tag{2.10}$$

Given this definition of the prepotential functions, and using (2.7) and the fact that the Kähler two form is closed, one can show

$$g_{ij*}(k^i_k k^{j*}_l - k^i_l k^{j*}_k) = f^i_{kl}\mathcal{P}_i\,. \tag{2.11}$$

This concludes our discussion of the vector multiplets. The above informations are sufficient to write down the explicit lagrangian, which is given in [11].

Hyper multiplets. For partial breaking of $\mathcal{N}=2$ supersymmetry it is necessary that that that the vectors are coupled to at least one hypermultiplet [9]. The reason for this is that when $\mathcal{N}=2$ is broken to $\mathcal{N}=1$ the massive gravitino is part of a massive spin 3/2 multiplet which contains two massive spin one fields. Thus the U(1), corresponding to the graviphoton, and the overall U(1) should become massive by Higgs mechanism. As the scalars of the vectors are in the adjoint representation, there must be at least two scalars from the hypermultiplet sector which are eaten up by two U(1)'s. Thus we have the condition that there must be at least one hypermultiplet and at least two U(1) translational isometries of the hypermultiplet manifold which provide the Higgs mechanism. To be specific about the hypermultiplet manifold, we choose to work with the minimal model, which has a single hypermultiplet parametrizing the coset $\mathrm{SO}(4,1)/\mathrm{SO}(4)$[1] using coordinates $b^u, u=0,1,2,3$. As $\mathrm{SO}(4,1)/\mathrm{SO}(4)$ is euclidean AdS_4 the metric on this quaternionic manifold can be chosen to b $h_{uv}=(\frac{1}{b^0})^2\delta_{uv}$. We choose the SU(2) connection ω^x on the quaternionic manifold as

$$\omega^x = \omega^x_u db^u\,, \qquad \omega^x_u = \frac{1}{b^0}\delta^x_u \tag{2.12}$$

where $x=1,2,3$. The quaternionic potentials is given by the field strength of the SU(2) connections, they are given by

$$\begin{aligned}\Omega^x &= \Omega^x_{uv}db^u\wedge db^v = d\omega^x + \frac{1}{2}\epsilon^{xyz}\omega^y\omega^z\,,\\ \Omega^x_{0u} &= -\frac{1}{2(b^0)^2}\delta^x_u\,, \qquad \Omega^x_{yz} = \frac{1}{2(b^0)^2}\epsilon^{xyz}\,.\end{aligned} \tag{2.13}$$

To gauge the hypermultiplet manifold we need to introduce the corresponding Killing vectors. We choose to gauge the hypermultiplet manifold by two translations corresponding to the U(1) of the graviphoton and the overall U(1) of the structure group $\mathrm{U}(N)$. The Killing vectors and the triplet of prepotentials $\mathcal{P}^x_\Lambda$ generating them, are given as follows

$$\begin{aligned}k^u_0 &= g_1\delta^{u1}+g_2\delta^{u2}\,, \qquad k^u_n = g_3\delta^{u2}\,,\\ \mathcal{P}^x_0 &= \frac{1}{b^0}(g_1\delta^{x1}+g_2\delta^{x2})\,, \qquad \mathcal{P}^x_n = g_3\frac{1}{b^0}\delta^{x2}\,, \qquad x=1,2,3\,.\end{aligned} \tag{2.14}$$

It is easy to check that the prepotentials and the Killing vectors satisfy the Poisson bracket condition for abelian gauge groups, given by

$$\Omega^x_{uv}k^u_0 k^v_n - \frac{1}{2}\epsilon^{xyz}\mathcal{P}^y_0\mathcal{P}^z_n = 0\,. \tag{2.15}$$

[1] In the next section we will generalize our discussion to the manifold $\mathrm{U}(M,2)/\mathrm{U}(M)\times\mathrm{U}(2)$.

To summarize, we have the following field content, $\mathcal{N}=2$ gravity multiplet, $\mathcal{N}=2$ vector multiplet gauged under the group $\mathrm{U}(N)$ and a hypermuliplet charged under the $\mathrm{U}(1)$ corresponding to the graviphoton and the overall $\mathrm{U}(1)$ of $\mathrm{U}(N)$. For future reference we write down the general form for the scalar potential which arises due to the gaugings of the vector multiplet manifold and the hypermultiplet manifold

$$V(z,\bar{z},b) = g^2\left[\left(g_{ab*}k^a_\Lambda k^{b*}_\Sigma + 4h_{uv}k^u_\Lambda k^v_\Sigma\right)\bar{L}^\Lambda L^\Sigma + g_{ab^*}f^\Lambda_a f^\Sigma_{b*}\mathcal{P}^x_\Lambda\mathcal{P}^x_\Sigma - 3\bar{L}^\Lambda L^\Sigma\mathcal{P}^x_\Lambda\mathcal{P}^x_\Sigma\right]. \tag{2.16}$$

Here $L^\Lambda = e^{K/2}X^\Lambda$ and $f^\Lambda_a = (\partial_a + \frac{1}{2}\partial_a K)L^\Lambda$. Note that every term in the scalar potential depends either on the moment maps or the Killing vectors.

2.2 Planck mass scalings

To obtain the rigid limit we need to take the Planck mass m_p to infinity. In this section we detail the Planck mass scalings of all the quantities involved in the $\mathcal{N}=2$ action, so that we can obtain the effective field theory in rigid limit. The fields and the coordinates of the $\mathcal{N}=2$ supergravity action as written in [11] are all dimensionless. To restore the canonical dimensions of the coordinates they are scaled as $x^\mu \to m_p x^\mu$, while the supersymmetry parameter or the superspace coordinate is scaled as $\epsilon^A \to \sqrt{m_p}\epsilon^A$. We follow a similar procedure for all the quantities in the lagrangian. First we discuss the scalings involved in the geometry, for the function f in (2.1), following [10] we assume the following scaling form

$$f(z) = \frac{1}{2} + \frac{\Lambda}{m_p}z^n + \frac{\Lambda^2}{m_p^2}\phi(z) \tag{2.17}$$

here the scale Λ will be shown to be the supersymmetry breaking scale subsequently, ϕ is an $\mathrm{U}(N)$ invariant function of z^i. Note that the linear term is a function only of z^n. The sections X^i and F_i are scaled as

$$X^i \to \frac{\Lambda}{m_p}X^i\,, \qquad F_i \to \frac{m_p}{\Lambda}F_i\,. \tag{2.18}$$

With these scalings for the holomorphic sections and f given by (2.17) we write down the scaling form of the following geometric quantities, which will be repeatedly used in our analysis

$$\begin{aligned}
\partial_i K &= -\frac{\Lambda^2}{m_p^2}\left(\frac{1}{2}(\partial_i\phi + \partial_{i^*}\phi^*) - \frac{1}{2}(z^a - z^a)\partial_i\partial_a\phi\right) + O\left(\frac{\Lambda^3}{m_p^2}\right),\\
\partial_n K &= -\frac{\Lambda}{m_p} + O\left(\frac{\Lambda^2}{m_p^2}\right),\\
g_{ij^*} &= -\frac{\Lambda^2}{m_p^2}\frac{1}{2}\left(\partial_i\partial_j\phi + \partial_{i*}\partial_{j*}\phi^*\right) + O\left(\frac{\Lambda^3}{m_p^3}\right),\\
g_{in^*} &= -\frac{\Lambda^2}{m_p^2}\frac{1}{2}\left(\partial_i\partial_n\phi + \partial_{i*}\partial_{n*}\phi^*\right) + O\left(\frac{\Lambda^3}{m_p^3}\right),\\
g_{nn^*} &= \frac{\Lambda^2}{m_p^2}\left(1 - \frac{1}{2}\partial_n\partial_n\phi - \frac{1}{2}\partial_{n*}\partial_{n^*}\phi^*\right) + O\left(\frac{\Lambda^3}{m_p^3}\right).
\end{aligned} \tag{2.19}$$

Note that the scaling behaviour of $\partial_i K$ and the Kähler metric is consistent with equation (2.11) relating the Killing vectors to the prepotential. The couplings of the hypermultiplet to the U(1)'s are scaled as follows

$$g_1 = \frac{\Lambda^2}{m_p^2}\xi\,, \qquad g_2 = \frac{\Lambda^2}{m_p^2}e\,, \qquad g_3 = 2\frac{\Lambda}{m_p}m\,. \tag{2.20}$$

We now go over to discuss the scalings of the fields. We scale the $\mathcal{N}=2$ gravity fields so as to retain a non-dynamical gravity background while the gravity fluctuations are scaled so that they decouple from the matter sector in the $m_p \to \infty$ limit. Thus the resulting theory has only global supersymmetry, but it is coupled to the $\mathcal{N}=2$ gravity backgrounds. The fields of the gravity multiplet scale as

$$\begin{aligned} V_\mu^a &\to V_\mu^{a(b)} + \frac{1}{m_p}\delta V_\mu^a\,, \qquad T_{\mu\nu} \to \frac{1}{m_p}T_{\mu\nu}^{(b)} + \frac{1}{m_p^2}\delta T_{\mu\nu}\,, \\ \Psi_{A\mu} &\to \frac{1}{\sqrt{m_p}}\Psi_{A\mu}^b + \frac{1}{m_p^{3/2}}\delta\Psi_{A\mu}\,. \end{aligned} \tag{2.21}$$

Here V_μ^a stands for the vielbein, $T_{\mu\nu}$ refers to the graviphoton field strength and $\Psi_{A\mu}$ refer to the two gravitinos of the $\mathcal{N}=2$ gravity multiplet. Note that the ratio of the fluctuations to the background is suppressed by $1/m_p$. As a simple example for the decoupling of the matter sector from the fluctuations of the supergravity field, consider a minimally coupled scalar whose kinetic term is canonically normalized. It is easy to see then that the interaction with the fluctuation of the metric is down by a factor of $1/m_p^2$.

In the matter sector, the vector multiplets scale as

$$G_{\mu\nu}^a \to \frac{1}{\Lambda m_p}G_{\mu\nu}^a\,, \qquad \lambda_A^a \to \frac{1}{\Lambda\sqrt{m_p}}\lambda_A^a \tag{2.22}$$

where $G_{\mu\nu}^a$ is the gauge field strength and λ_A^a are the two gauginos. These scalings ensure that the kinetic terms for the gauginos and for the gauge fields are canonically normalized. We do have to scale also the scalars of the vector multiplets by $z \to z/\Lambda$, in order to canonically normalize their kinetic term, however for convenience we will reserve this for section 3.1. However there are terms in the lagrangian which couple the gravitational backgrounds of (2.21) to the matter sector and survive the $m_p \to \infty$ limit. Though we will not keep track of these terms explicitly, we will incorporate them in the analysis of chiral ring equations in section 5.

Finally the hypermultiplets scalings are given by

$$b^0 \to b^{0(b)} + \frac{1}{m_p}\delta b^0\,, \qquad b^x \to \frac{1}{m_p}b^x\,, \qquad \xi^\alpha \to \frac{1}{m_p^{3/2}}\xi^\alpha\,. \tag{2.23}$$

Note that the couplings of the hypermultiplets to the vectors is dictated by the charges which are scaled as in (2.20). This and the above scaling for the hypermultiplets ensures that b^x and δb^0 decouples from the dynamics. $b^{0(b)}$ is the background value which enters as a coupling constant in the decoupled theory with global supersymmetry.

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3. Partial breaking of $\mathcal{N}=2$ in the rigid limit

In this section we look for vacua with $\mathcal{N}=1$ supersymmetry in the rigid limit. In the rigid limit since fluctuations of the graviton and the hyperino decouple it is sufficient to study the supersymmetry transformation of the gauginos and look for vacua which preserve a single supersymmetry. As we mentioned before we will not keep track of the gravitational background couplings to the resulting $\mathcal{N}=1$ theory in the lagrangian but will include them in the analysis of the chiral ring relations in section 5. In fact including the gravitational background couplings makes it extremely cumbersome to write a compact form for the lagrangian, without them can summarize the resulting lagrangians compactly in $\mathcal{N}=1$ superspace.

To analyze the supersymmetry of the vacuum we study the gaugino shifts which are given by

$$\begin{aligned}\delta\lambda^{Aa} &= W^{aAB}\eta_B + \cdots \\ &= \left(\epsilon^{AB}k^a_\Lambda \bar{L}^\Lambda + i(\sigma_x)_C{}^B\epsilon^{CA}\mathcal{P}^x_\Lambda g^{ab*}\bar{f}^\Lambda_{b*}\right)\eta_B + \cdots\,.\end{aligned} \tag{3.1}$$

There are other terms in the supersymmetric variation of the gauginos which represent the dots. But the terms explicitly written down in (3.1) are alway present irrespective of which gravitational backgrounds are turned on, therefore they dictate the number of supersymmetry preserved at various vacua. Substituting the scalings of all the fields in the above expression we obtain

$$\begin{aligned}\delta\lambda^{Ai} &= \Lambda^2\epsilon^{AB}\frac{1}{\sqrt{2}}f^i_{jk}z^k\bar{z}^j\eta_B + i\Lambda^2\sqrt{2}\epsilon_{BC}\frac{1}{b^0}\tau_2^{ia}\left(\xi\sigma^{1C}_A\delta_{a*n*} + (e\delta_{a*n*}+m\bar{\tau}_{a*n*})\sigma^{2C}_A\right)\eta^B + \\ &\quad + O\left(\frac{\Lambda^3}{m_p}\right), \\ \delta\lambda^{An} &= i\Lambda^2\sqrt{2}\epsilon_{BC}\frac{1}{b^0}\tau_2^{na}\left(\xi\sigma^{1C}_A\delta_{a*n*} + (e\delta_{a*n*}+m\bar{\tau}_{a*n*})\sigma^{2C}_A\right)\eta^B + O\left(\frac{\Lambda^3}{m_p}\right),\end{aligned} \tag{3.2}$$

where in $b^{(0)}$ we have suppressed the superscript to indicate the background and we have defined

$$\begin{aligned}\mathcal{F}(z) &= i(z^n)^2 - 2i\phi(z)\,, \\ \tau_{ij}(z) &= \tau_1(z)_{ij} + i\tau_2(z)_{ij} = \partial_i\partial_j\mathcal{F}(z)\,.\end{aligned} \tag{3.3}$$

Now it is easy to see that we have a vacuum with $\mathcal{N}=1$ supersymmetry. Consider the following vacuum values

$$z^i = 0\,, \qquad \tau_{ij} = \left(-\frac{e}{m} + i|\frac{\xi}{m}|\right)\delta_{ij}\,, \qquad \tau_{in} = 0\,, \qquad \tau_{nn} = -\frac{e}{m} + i|\frac{\xi}{m}|\,. \tag{3.4}$$

For these values of the scalar field the shifts on the gauginos reduce to

$$\begin{aligned}\delta\lambda^{Ai} &= 0\,, \\ \delta\lambda^{An} &= i\Lambda^2\sqrt{2}\epsilon_{BC}\frac{1}{b^0}m(\sigma^1+i\sigma^2)^C_A\eta^B\,.\end{aligned} \tag{3.5}$$

From this it is clear for the vacuum (3.4) there is one unbroken supersymmetry as the shift on the gaugino λ^{An} has one zero eigen value. Also note that the scale of the supersymmetry breaking is controlled by Λ. Due to the gauging of the hypermultiplet manifold and the vector multiplet manifold, there is a potential for the scalars of the vector multiplet which is given by (2.16). The leading term in the potential which will contribute in the $m_p \to \infty$ limit is given by

$$V(z) = g^2 \frac{\Lambda^4}{m_p^4}\left(\frac{1}{4}\tau_{2ij} f^i_{lm}\bar{z}^l z^m f^j_{np} z^n \bar{z}^p + \frac{1}{b^{02}}\tau_2^{nn}\xi^2 + \right.$$
$$\left. + \frac{1}{b^{02}}\tau_2^{ab*}(e\delta_{an} + m\tau_{an})(e\delta_{b*n*} + m\tau_{b*n*})\right) - \frac{\Lambda^4}{m_p^4}\frac{1}{2}\left(\xi^2 + e^2 + 4m^2\right). \quad (3.6)$$

The m_p dependence in the scalar potential cancel in the lagrangian as the coordinate x is scaled to $x \to m_p x$. The first term in the scalar potential arises from the first term of (2.16), it corresponds to the scalar potential arising due to the D-term of a rigid $\mathcal{N} = 1$ theory. The second and third terms of (3.6) are due to the coupling of the vectormultiplet manifold to the gaugings of the hypermultiplet manifold. These terms are obtained by substituting the apropriate scalings in the third term of the potential in (2.16). The constant contribution to the scalar potential in (3.6) in obtained from the rigid limit of the second and the last term of (2.16). It is also easy to see that the the vacuum (3.4) is the minimum of the above scalar potential. The minimum value of the potential is infact not zero but is given by

$$V_{\min} = g^2 \frac{\Lambda^2}{2b^{02}}(4\xi m - \xi^2 - e^2 - 4m^2)\,, \quad (3.7)$$

thus it is clear that the vacuum (3.4) does not preserve $\mathcal{N} = 1$ supersymmetry in the fluctuations of the gravitational sector which are decoupled. However the gravitational backgrounds do transform into each other under the full $\mathcal{N} = 2$ supersymmetry transformations. This is because the constants shifts of the gravitinos are suppressed with respect to the variation of the backgrounds by $1/m_p$, which is the same ratio of scales between the background and the fluctuations. In the subsequent discussion, as we will be interested in the rigid limit we will, neglect the zero point energy (3.7) and will set the background moduli $b^0 = 1$.

3.1 Structure of the theory about the $\mathcal{N} = 1$ vacuum

To study the structure of the low energy theory which is obtained at the $\mathcal{N} = 1$ vacuum we will first evaluate the mass matrices for the fermions and show that half the fermions are massless and half of them are massive. Thus the gaugino of the $\mathcal{N} = 2$ vector multiplet splits into the partner of the $\mathcal{N} = 1$ gauge multiplet and a massive $\mathcal{N} = 1$ chiral multiplet in the adjoint representation. The masses of the fermions are extracted from the term $\mathcal{M}_{aAbB}\lambda^{\bar{a}A}\lambda^{bB}$ where

$$\mathcal{M}_{aAbB} = \frac{1}{3}\left(\epsilon_{AB} g_{aa*} k^{a*}_\Lambda f^\Lambda_b + i\sigma^C_{xA}\epsilon_{CB}\mathcal{P}^x_\Lambda \nabla_a f^\Lambda_b\right). \quad (3.8)$$

The $\mathcal{N} = 1$ vacuum preserves the U(N) gauge symmetry thus the expectation valued $\langle z^i\rangle = 0$, which enables us to drop the first term in (3.8). To evaluate the second term in the mass matrix we need the Christoffel symbols of the vector multiplet manifold at the $\mathcal{N}=1$ vacuum to the leading order in Λ/m_p. Using the scaling in (2.19), the non vanishing Christoffel symbols to the leading order in Λ/m_p are

$$\Gamma^n_{ab} = -\tau_2^{nn}\partial_a\partial_b\partial_n\phi\,. \tag{3.9}$$

Substituting this and using the scaling of (2.17) and (2.20) we obtain

$$\mathcal{M}_{aAbB} = -\frac{\Lambda^3}{m_p^3}\frac{1}{6\sqrt{2}}\,\epsilon_{AC}m(\sigma^1+i\sigma^2)^C_B\partial_a\partial_b\partial_n\phi\big|_{z^a=0}\,. \tag{3.10}$$

The m_p scaling of the fermion mass is such that the mass term $\mathcal{M}_{aAbB}\bar{\lambda}^{aA}\lambda^{bB}$ scales like Λ/m_p^4, which is the right scaling so that it survives in the $m_p\to\infty$ limit. The fermion mass is proportional to Λ, the supersymmetry breaking scale as expected. Now it is easy to see that this mass matrix has N^2 zero eigenvalues, the corresponding gauginos being members of the $\mathcal{N}=1$ gauge multiplets. The N^2 non-zero eigenvalues, give mass proportional to $\partial_a\partial_b\partial_n\phi$ to the fermions which are part of the $\mathcal{N}=1$ chiral multiplets. Thus from (3.10) we see that the fermions λ^{i1} are the gauginos and λ^{i2} are the fermions of the chiral multiplets which become massive.

The structure of the $\mathcal{N}=1$ theory will be dictated by the superpotential for the chiral multiplets and the gauge function. Since the $\mathcal{N}=1$ theory is obtained by spontaneously breaking $\mathcal{N}=2$ the gauge function and the superpotential are closely related, both of them are derived from a prepotential. We will see how this happens for the theory we are discussing at the $\mathcal{N}=1$ vacuum. We organize the expansion of the scalar potential in (3.6) around the the vacuum in (3.4) using the following form for the expansion of the gauge function

$$\begin{aligned}\tau_{ij} &= \left(-\frac{e}{m}+i|\frac{\xi}{m}|\right)\delta_{ij}+\tau'_{ij}\,,\\ \tau_{in} &= \partial_i\mathcal{W}\,,\\ \tau_{nn} &= -\frac{e}{m}+i|\frac{\xi}{m}|+\partial_n\mathcal{W}\,.\end{aligned} \tag{3.11}$$

This parametrization of τ_{in} and τ_{nn} can always be done because of its definition (3.3) with $\mathcal{W}=\partial_n\mathcal{F}$. Substituting this form into the expression for the scalar potential in (3.6) we obtain

$$V(z) = g^2\frac{\Lambda^4}{m_p^4}\left(\frac{1}{4}\tau_{2ij}f^i_{lm}\bar{z}^lz^mf^j_{np}z^n\bar{z}^p+m^2\tau_2^{ab*}\partial_a\mathcal{W}(\partial_b\mathcal{W})^*\right)+V_{\min}\,. \tag{3.12}$$

Before we go ahead we will fix the factor g that appears in the scalar potential. From [11] we see that covariant derivatives of the scalar fields z are given by

$$\begin{aligned}dz^i+gA^\Lambda k^i_\Lambda(z) &= dz^i+gf^\Lambda_kA^kf^i_{\Lambda,j}z^j+\cdots\\ &= \frac{1}{m_p}\left(dz^i+g\frac{1}{\sqrt{2}}A^kf^i_{kj}z^j+\cdots\right).\end{aligned} \tag{3.13}$$

Here we have expressed the field A^Λ in terms of the gauge fields A^i and the graviphoton which are represented by the dots. In writing this equality we have ignored the contribution to the covariant derivative from the graviphoton as the first term is sufficient to fix the factor g. The second equation in (3.13) is obtained by substituting the required scalings, now it is clear that $g = \sqrt{2}$ is a convenient choice so that we obtain the standard commutator term without extra factors.

We now detail the structure of the $\mathcal{N} = 1$ theory. From the scalar potential we see that the $\mathcal{N} = 1$ theory has a superpotential $m\mathcal{W}$, a nontrivial gauge function τ and a nontrivial Kähler term with a non trivial metric for the scalars also given by τ. To arrive at the $\mathcal{N} = 1$ theory with a polynomial superpotential we choose the following prepotential.

$$\begin{aligned} \mathcal{F} &= \left(-\frac{e}{m} + i|\frac{\xi}{m}|\right) \operatorname{Tr}\frac{(z^2)}{2} + \sum_{k=1}^{p} \frac{1}{(k+1)(k+2)} g'_k \operatorname{Tr}(z^{k+2}) \\ &= \left(-\frac{e}{m} + i|\frac{\xi}{m}|\right) \operatorname{Tr}\frac{(z^2)}{2} + \mathcal{F}' . \end{aligned} \tag{3.14}$$

Here z refers to the $N\times N$ matrix given by $z = \frac{z^n}{\sqrt{N}}I + z^i T_i$. We have chosen the prepotential so that it satisfies the condition that, at $z^a = 0$, one has the required vacuum values as in (3.4). Thus from comparison of the expansion in (3.11) and the scalar potential in (3.12) the superpotential is given by

$$\begin{aligned} \mathcal{W} &= m\partial_n \mathcal{F}' = \frac{m}{\sqrt{N}} \sum_{k=1}^{p} \frac{1}{k+1} g_k \operatorname{Tr}(z^{k+1}) , \\ &= \tilde{m} \sum_{k=1}^{p} \frac{1}{k+1} g_k \operatorname{Tr}(z^{k+1}) . \end{aligned} \tag{3.15}$$

In the second equality we have absorbed a factor of $\sqrt{N}$ by the redefinition $m = \tilde{m}\sqrt{N}$ for convenience. The gauge kinetic function is given by

$$\begin{aligned} \tau_{ij} &= \left(-\frac{e}{m} + i|\frac{\xi}{m}|\right) \delta_{ij} + \partial_{ij}\mathcal{F}' , \\ \tau_{in} &= \partial_{in}\mathcal{F}' , \\ \tau_{nn} &= -\frac{e}{m} + i|\frac{\xi}{m}| + \partial_{nn}\mathcal{F}' . \end{aligned} \tag{3.16}$$

Again from the expansion of the scalar potential we see that the Kähler metric for the scalars is given by $\tau_{2ij}/2$

Finally, before writing the resulting action in superspace we will restore the canonical dimension of the scalar field z by scaling $z \to z/\Lambda$ and as a result given dimensions to the couplings in the superpotential. To extract the canonical dimension of the superpotential, one scales $\mathcal{W} \to \mathcal{W}/\Lambda^3$. Doing this would require the scaling of the couplings $g_k \to \Lambda^{k-2} g_k$, this ensures that that the scaled superpotential is given by (3.15) with the understanding that the field z and the couplings have the appropriate dimensions. However, the gauge kinetic function contains one more derivative with respect to z, and therefore performing the same scaling of z and the couplings in the gauge kinetic function we obtain the following scaling forms

$$\begin{aligned}
\tau_{ij} &= \left(-\frac{e}{m} + i|\frac{\xi}{m}|\right)\delta_{ij} + \frac{1}{\Lambda^2}\partial_{ij}\mathcal{F}' , \\
\tau_{in} &= \frac{1}{\Lambda^2}\partial_{in}\mathcal{F}' , \\
\tau_{nn} &= -\frac{e}{m} + i|\frac{\xi}{m}| + \frac{1}{\Lambda^2}\partial_{nn}\mathcal{F}' .
\end{aligned} \tag{3.17}$$

Note that if one is interested in the dynamics of the theory at scales much smaller that the supersymmetry breaking scale Λ we obtain a $\mathcal{N}=1$ theory with a trivial gauge kinetic function and a trivial Kähler metric for the scalars leading to theory discussed in [1]. We summarize the $\mathcal{N}=1$ theory obtained from partial breaking of $\mathcal{N}=2$ in superspace.

$$\begin{aligned}
S &= -\frac{i}{4}\int d^2\theta\tau_{ab}(\Phi)W^aW^b + \int d^2\theta\mathcal{W}(\Phi) + S_{\text{Kahler}} , \\
S_{\text{Kahler}} &= \int d^4\theta K(\Phi,\Phi^\dagger) , \\
K(z,\bar{z}) &= \frac{i}{4}\left(z^a\partial_{a*}\bar{\mathcal{F}} - \bar{z}^a\partial_a\mathcal{F}\right) .
\end{aligned} \tag{3.18}$$

Here the Kähler potential is of dimension 2 and we have used the scaled z and the scaled couplings. This is the non-abelian generalization of the $\mathcal{N}=1$ model obtained by [21] for partial breaking in rigid $\mathcal{N}=2$ theories.

4. Partial breaking with hypermultiplets in the rigid limit

As we have seen in the previous section we need at least one hypermultiplet in the hidden sector in order to break $\mathcal{N}=2$ supergravity. The low energy theory in the rigid limit was $N=1$ with an adjoint chiral multiplet. In this section we would like to generalize the discussions of the previous section to obtain chiral multiplets charged in the fundamental representation of the gauge group in the rigid limit of spontaneously broken $\mathcal{N}=2$ supergravity. We will discuss the case of obtaining chiral multiplets charged in the fundamental of $\mathrm{U}(N)$ in detail and then outline the case for other representations.

We discuss the field content and the gaugings necessary in detail. We start with a $n+1$ dimensional special Kähler manifold for the vector multiplet. The holomorphic sections on this manifold is given by

$$\begin{aligned}
X^0(z) &= \frac{1}{\sqrt{2}}, & F_0(z) &= -\frac{i}{\sqrt{2}}\left(2f(z) - z^a\frac{\partial f(z)}{\partial z^a}\right), \\
X^i(z) &= \frac{z^i}{\sqrt{2}} & F_i(z) &= -\frac{i}{\sqrt{2}}\frac{\partial f(z)}{\partial z^i} , \\
X^{n+1}(z) &= \frac{i}{\sqrt{2}}\frac{\partial f(z)}{\partial z^{n+1}} , & F_{n+1}(z) &= \frac{z^{n+1}}{\sqrt{2}} .
\end{aligned} \tag{4.1}$$

Note that, other than for the inclusion of an additional coordinate, this holomorphic sections is similar to (2.1). Here i,j take values from $1,\dots n$, and a takes values from 1 to

$(n+1)=m$. The i,j indices parametrize the $\mathrm{U}(N)$ directions, with $n=N^2$, while the m th direction corresponds to an additional $\mathrm{U}(1)$. Thus we start with the gauge group $\mathrm{U}(N)\times\mathrm{U}(1)$. We need this additional $\mathrm{U}(1)$ as it is not possible to obtain hypers with respect to the single $\mathrm{U}(1)$ for $\mathcal{N}=1$ obtained by partially breaking $\mathcal{N}=2$ [22]. Thus we need an additional $\mathrm{U}(1)$ to provide $\mathrm{U}(1)$ charges for the hypers. In the spontaneously broken theory the $\mathrm{U}(1)$ responsible for the gauging which partially breaks the supersymmetry will be decoupled. The action of the group $\mathrm{U}(N)$ on the coordinates z^i is the given by $\mathrm{U}(z^iT_i)U^\dagger$, here T_i stand for the the $N\times N$ hermitean matrices which generate the full group $\mathrm{U}(N)$. The section (4.1) satisfy the special Kähler condition given in (2.3). The Kähler potential and the Killing vectors are given by similar equations as (2.4) and (2.5), except that the structure group is $\mathrm{U}(N)$ and the coordinates i,j,k run from 1 to N.

To obtain matter in the fundamental representation of $\mathrm{U}(N)$ it is convenient to take the hypermultiplet manifold to be the following homogeneous, symmetric, quaternionic manifold

$$\mathcal{M}=\frac{\mathrm{U}(2,N+1)}{\mathrm{U}(2)\times\mathrm{U}(N+1)}\,. \tag{4.2}$$

This manifold has dimension $4(N+1)$. Indeed we need one additional hypermultiplet to play the role of $\mathrm{SO}(4,1)/\,\mathrm{SO}(4)$ of the previous section. this is the minimal requirement for spontaneous breaking of $\mathcal{N}=2$ supergravity, however here the hypermultiplet is non-trivially embedded in $\mathcal{M}$. We parametrize this coset manifold using the following $(3+N)\times(3+N)$ matrix

$$\mathcal{L}(q)=\begin{pmatrix}\sqrt{1+qq^\dagger} & q\\ q^\dagger & \sqrt{1+q^\dagger q}\end{pmatrix}=\begin{pmatrix} r_1 & q\\ q^\dagger & r_2\end{pmatrix} \tag{4.3}$$

here q is a $2\times(N+1)$ matrix with complex entries. We label the entries as q^{Au}, A running over $1,2$ and u running over $1,\dots m$. Note that this parametrization ensures that $\mathcal{L}$ satisfies

$$\begin{aligned}\mathcal{L}\Omega\mathcal{L}^\dagger&=\Omega\,,\\ \Omega&=\begin{pmatrix}-1_{2\times2} & 0\\ 0 & 1_{(N+1)\times(N+1)}\end{pmatrix}.\end{aligned} \tag{4.4}$$

From this parameterization, the coset vielbein E, the $\mathrm{U}(2)$ connection θ and the $\mathrm{U}(N+1)$ connection Δ are given by

$$\begin{aligned}E&=r_1dq-qdr_2=dq-\frac{1}{2}qdq^\dagger dq+O(q^4)\,,\\ \theta&=r_1dr_1-qdq^\dagger=\frac{1}{2}(dqq^\dagger-qdq^\dagger)+O(q^3)\,,\\ \Delta&=r_2dr_2-q^\dagger dq=\frac{1}{2}(dq^\dagger q-q^\dagger dq)+O(q^3)\,.\end{aligned} \tag{4.5}$$

The metric on the quaternionic manifold $\mathcal{M}$ is given by

$$ds^2=E^\dagger\otimes E=dq^\dagger dq-\frac{1}{2}\left(q^\dagger dqq^\dagger dq+dq^\dagger qdq^\dagger q\right)+O(q^4)\,. \tag{4.6}$$

We have written down the expansion in powers of q as it will be useful for us later, in taking the $m_p \to \infty$ limit. The HyperKähler 2-form K^x and the SU(2) triplet 1-forms are given by

$$K^x = \frac{1}{2}\,\mathrm{Tr}(E^\dagger \wedge \sigma^x E)\,, \qquad w^x = -\frac{1}{2}\,\mathrm{Tr}(\theta\sigma^x) \tag{4.7}$$

where σ^x refers to Pauli spin matrices. We denote the vielbein 1-form as $\mathcal{U}^{Aai}$, where we have split the Sp$(2N+2)$ indices as ai with $a = 1, 2$ and $i = 1, \dots N+1$. The components of the vielbein 1-form are defined by

$$\begin{aligned} \mathcal{U}^{Aai} &= \mathcal{U}^{Aai}_{Bu} dq^{Bu} + \mathcal{U}^{Aai}_{B\bar{u}} d\bar{q}^{B\bar{u}}\,, \\ &= \delta^{a1}\epsilon^{AB}\delta^{ij}E^{Bj} + \delta^{a2}\delta^{AB}\delta^{ij}E^{\dagger jB}\,. \end{aligned} \tag{4.8}$$

We now write down the Killing vectors corresponding to the isometries of the manifold $\mathcal{M}$. For every element $g \in u(2, N+1)$, the Lie algebra of U$(2, N+1)$, the following action is an isometry

$$\delta\mathcal{L} \equiv \left(k_g^{Au}\frac{\partial}{\partial q^{Au}} + k_g^{A\bar{u}}\frac{\partial}{\partial q^{A\bar{u}}}\right)\mathcal{L} = g\mathcal{L} - \mathcal{L}w_g \tag{4.9}$$

where $g \in u(2, N+1)$. $w_g \in u(2) \oplus u(m)$ is the right-compensator, which depends on g and is needed to keep $\delta\mathcal{L}$ in the form of (4.3). g and w_g are parametrized by

$$g = \begin{pmatrix} a & b \\ b^\dagger & c \end{pmatrix} \in u(2, N+1)\,, \qquad w_g = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix} \in u(2) \otimes u(N+1) \tag{4.10}$$

with a and b anti-hermitian matrices. Expanding (4.9) out we get

$$\begin{aligned} \delta q &= aq + br_2 - qc - q\hat{w}_2 = aq + r_1 b - qc + \hat{w}_1 q\,, \\ w_1 &= q + \hat{w}_1\,, \qquad w_2 = c + \hat{w}_2\,. \end{aligned} \tag{4.11}$$

To leading order the solutions for $\hat{w}$'s in terms of g is is given by

$$\hat{w}_1 = \frac{1}{2}(bq^\dagger - qb^\dagger) + O(q^3)\,, \qquad \hat{w}_2 = \frac{1}{2}(b^\dagger q - q^\dagger b) + O(q^3)\,. \tag{4.12}$$

Using all this killing vectors to the leading order in q for a given g is given by

$$\begin{aligned} \vec{k}_a &= (aq)^{Au}\frac{\partial}{\partial q^{Au}} + (q^\dagger a^\dagger)^{\bar{u}A}\frac{\partial}{\partial q^{\dagger A\bar{u}}}\,, \\ \vec{k}_a &= -(qc)^{Au}\frac{\partial}{\partial q^{Au}} - (c^\dagger q^\dagger)^{\bar{u}A}\frac{\partial}{\partial q^{\dagger A\bar{u}}}\,, \\ \vec{k}_a &= (b)^{Au}\frac{\partial}{\partial q^{Au}} + (b^\dagger)^{\bar{u}A}\frac{\partial}{\partial q^{\dagger A\bar{u}}}\,, \end{aligned} \tag{4.13}$$

where we have used the subscripts to denote the entry of g. Therefore Killing vector for the group element g is given by $\vec{k}_g = \vec{k}_a + \vec{k}_c + \vec{k}_b$. From the Killing vectors is easy to compute the moment map, which is defined by the equation

$$\mathbf{i}_{\vec{k}_g} K^x = -\nabla\mathcal{P}^x\,. \tag{4.14}$$

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From the fact that the curvature of the SU(2) connection is proportional to the HyperKähler structures, we can solve the above equation by the ansätz

$$\mathcal{P}^x_g = \frac{1}{2}\,\mathrm{Tr}\left[\begin{pmatrix} \sigma^x & 0 \\ 0 & 0_{(N+1)\times(N+1)} \end{pmatrix} L^{-1} g L\right]. \tag{4.15}$$

For later convenience we write down the leading contribution to the moment maps from a, b and c of the isometry g.

$$\begin{aligned}
\mathcal{P}^x_a &= \frac{1}{2}\,\mathrm{Tr}\left(\sigma^x a + \frac{\sigma^x}{2}(qq^\dagger a + aqq^\dagger)\right), \\
\mathcal{P}^x_b &= -\frac{1}{2}\,\mathrm{Tr}\left(\sigma^x q b^\dagger - \sigma^x b q^\dagger\right), \\
\mathcal{P}^x_c &= -\frac{1}{2}\,\mathrm{Tr}\left(\sigma^x q c q^\dagger\right).
\end{aligned} \tag{4.16}$$

We now give the two commuting matrices, which belong to the Lie algebra $u(2, N+1)$, which are the generators of R^2. We will gauge along this isometries to break supersymmetry. These Killing directions function as the two translation isometries of the minimal hypermultiplet manifold $\mathrm{SO}(4,1)/\mathrm{SO}(4)$ that we used in the previous section to break supersymmetry.

$$\begin{aligned}
g_0 &= i\begin{pmatrix}
\frac{\xi}{2} & i\frac{e}{2} - \frac{\xi}{2} & \frac{1}{2\sqrt{2}}(e + 2i\xi) & 0 & \cdots \\
-i\frac{e}{2} - \frac{\xi}{2} & \frac{\xi}{2} & \frac{1}{2\sqrt{2}}(e - 2i\xi) & 0 & \cdots \\
\frac{1}{2\sqrt{2}}(-e + 2i\xi) & \frac{1}{2\sqrt{2}}(-e - 2i\xi) & -\xi & 0 & \cdots \\
0 & 0 & 0 & & \\
\vdots & & & & MI_{N\times N}
\end{pmatrix} \\
g_{n+1} &= i\begin{pmatrix}
0 & im & \frac{m}{\sqrt{2}} & 0 & \cdots \\
-im & 0 & \frac{m}{\sqrt{2}} & 0 & \cdots \\
-\frac{m}{\sqrt{2}} & -\frac{m}{\sqrt{2}} & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & \cdots \\
\vdots & & & \vdots & \cdots
\end{pmatrix}.
\end{aligned} \tag{4.17}$$

Here the parameters ξ, e, m are real, and it is easy to see that these isometries commute. Both of them contain a non zero b component, thus they generate translations also, these isometries are used to break supersymmetry. The remaining isometries, which ensure that the hypermultiplets surviving the supersymmetry breaking in the rigid limit are gauged with respect to the $\mathrm{U}(N)$ gauge fields, are given by

$$g_i = \begin{pmatrix}
0\;0\;0\;0 & \cdots \\
0\;0\;0\;0 & \cdots \\
0\;0\;0\;0 & \cdots \\
0\;0\;0\;0 & iT^i_{N\times N}
\end{pmatrix} \tag{4.18}$$

here T^i are $N \times N$ hermitean generators of $\mathrm{U}(N)$.

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We provide the various scalings necessary in order to obtain $\mathcal{N}=1$ supersymmetry in the rigid limit. The scalings of the vector multiplet manifold are given by

$$\begin{aligned} f(z) &= \frac{1}{2}+\frac{\Lambda}{m_p}z^m+\frac{\Lambda^2}{m_p^2}\phi(z,z^m)\,, \\ X^i &\to \frac{\Lambda}{m_p}\,, \qquad F_i \to \frac{m_p}{\Lambda}F_i\,. \end{aligned} \tag{4.19}$$

The gaugings in (4.17) are scaled as follows

$$\xi \to \frac{\Lambda^2}{m_p^2}\xi\,, \qquad e \to \frac{\Lambda^2}{m_p^2}e\,, \qquad m \to \frac{\Lambda}{m_p}m\,, \qquad M \to \frac{\Lambda}{m_p}M\,. \tag{4.20}$$

The gravity multiplet and the vector multiplet scale as in (2.21) and (2.22) respectively The hypermultiplet fields scale as

$$q^u \to \frac{1}{m_p}q^u\,, \qquad \zeta^\alpha \to \frac{1}{m_p^{3/2}}\zeta^\alpha \tag{4.21}$$

where we denote by ζ the hyperinos, partners of the q's. The supersymmetry variation with these scalings are given by

$$\begin{aligned} \delta\lambda^{Ai} &= \Lambda^2\epsilon^{AB}\frac{1}{\sqrt{2}}f^i_{jk}z^k\bar{z}^j\eta_B - i\sqrt{2}\Lambda^2\epsilon_{CA}\tau_2^{ia}(\sigma_x)_C{}^B\left(\mathcal{P}_0^x\delta_{a*n*}+2\mathcal{P}_m^x\bar{\tau}_{a*n*}\right)\eta^B + O\left(\frac{\Lambda^3}{m_p}\right), \\ \delta\lambda^{An} &= -i\sqrt{2}\Lambda^2\epsilon_{CA}\tau_2^{na}(\sigma_x)_C{}^B\left(\mathcal{P}_0^x\delta_{a*n*}+2\mathcal{P}_m^x\bar{\tau}_{a*n*}\right)\eta^B + O\left(\frac{\Lambda^3}{m_p}\right). \end{aligned} \tag{4.22}$$

It is clear that we can still choose the vacuum values for the vector multiplet moduli to be given similar to (3.4).

$$z^i=0\,, \qquad \tau_{ij}=\left(-\frac{e}{m}+i|\frac{\xi}{m}|\right)\delta_{ij}\,, \qquad \tau_{im}=0\,, \qquad \tau_{mm}=-\frac{e}{m}+i\left|\frac{\xi}{m}\right|. \tag{4.23}$$

Substituting these values in the first equation of (4.22) we see that variation $\delta\lambda^{Ai}=0$. To see the reduction of supersymmetry down to $\mathcal{N}=1$, we consider the last second equation of (4.22), after substituting the values of the moment maps, we see that the only contribution comes from $\mathcal{P}_a^x$, the others being suppressed by $1/m_p$. The variation is given by

$$\delta\lambda^{An}=\sqrt{2}\Lambda^2\epsilon_{AC}m(\sigma^1+i\sigma^2)_C{}^B\eta^B\,. \tag{4.24}$$

Therefore we see now that this vacuum has one unbroken supersymmetry.

We expand the theory around this vacuum and write down the structure of the resulting $\mathcal{N}=1$ theory. The leading terms in the scalar potential, which survive in the rigid limit is given by.

$$\begin{aligned} V &= g^2(V_1+V_2+V_3+V_4)\,, \\ V_1 &= \frac{\Lambda^4}{m_p^4}\left(\frac{1}{4}\tau_{2ij}f^i_{lm}\bar{z}^l\bar{z}^m f^j_{np}z^n\bar{z}^p+\tau_2^{nn}\left(\frac{\xi}{2}\right)^2+\tau_2^{ab}\left(\frac{e}{2}\delta_{am}+\frac{m}{2}\tau_{am}\right)\left(\frac{e}{2}\delta_{bm}+\frac{m}{2}\bar{\tau}_{bm}\right)\right) \end{aligned}$$

$$
\begin{aligned}
V_2 = &= \frac{1}{m_p^4}\frac{1}{4}\tau_2^{ij}\,\mathrm{Tr}(\sigma^x q T_i q^\dagger)\,\mathrm{Tr}(\sigma^x q T_j q^\dagger) + \frac{\Lambda^2}{m_p^4}\frac{1}{2}\tau_2^{mi}\left(\xi\,\mathrm{Tr}(\sigma^1 q T_i q^\dagger) + e\,\mathrm{Tr}(\sigma^2 q T_i q^\dagger)\right) + \\
&\quad + \frac{\Lambda^2}{m_p^4}\frac{m}{4}\tau_2^{ai}\left(\tau_{ma} + \bar{\tau}_{ma}\right)\mathrm{Tr}(\sigma^2 q T_i q^\dagger) \\
V_3 &= \frac{\Lambda^2}{m_p^4}\left(2\,\mathrm{Tr}(q M^2 q^\dagger) + 2\,\mathrm{Tr}(q M T_i q^\dagger)(z^i + \bar{z}^i)\right) \\
V_4 &= -\frac{\Lambda^4}{m_p^4}(e^2 - 5\xi^2 + 4m^2)\,. \qquad (4.25)
\end{aligned}
$$

Note that the first hypermultiplet, q^{A1}, is massless, and it does not contribute to the scalar potential. Thus it plays the same role as the modulus b_0 in the discussion of the previous section. We now expand this potential around the minima (4.23), adopting the same form for the expansion of the gauge function as in (3.11)

$$
\tau_{ij} = \left(-\frac{e}{m} + i|\frac{\xi}{m}|\right)\delta_{ij} + \tau'_{ij}\,, \qquad \tau_{im} = \partial_i \mathcal{W}\,, \qquad \tau_{mm} = -\frac{e}{m} + i|\frac{\xi}{m}| + \partial_m \mathcal{W}\,, \quad (4.26)
$$

Substituting these expansions in the scalar potential (4.25) we obtain

$$
\begin{aligned}
V_1 &= \left(\frac{\Lambda}{m_p}\right)^4\left(\frac{1}{4}\tau_{2ij} f^i_{lm}\bar{z}^l z^m f^j_{np} z^n \bar{z}^p + 2m\xi + \left(\frac{m}{2}\right)^2 \tau_2^{ab*}\partial_a \mathcal{W}(\partial_b \mathcal{W})^*\right), \\
V_2 &= \frac{1}{m_p^4}\left(\frac{1}{4}\tau_2^{ij}(q^1 T_i q^2 - q^{2\dagger} T_j q^{1\dagger})^2 + (q^1 T_i q^2)(q^{2\dagger} T_j q^{1\dagger}) + \right. \\
&\qquad \left. + i\frac{m\Lambda^2}{2}\tau_2^{ji}\partial_j \mathcal{W}(q^{2\dagger} T_i q^{1\dagger}) - i\frac{m\Lambda^2}{2}\tau_2^{ji}(\partial_j \mathcal{W})^*(q^1 T_i q^2)\right), \\
V_3 &= \frac{\Lambda^2}{m_p^4}\Big(2M^2 q^{1i*}q^{1i} + 2M^2 q^{2i*}q^{2i} + 2(q^1 M T_i q^{1\dagger} + q^{2\dagger} M T_i q^2)(z^i + \bar{z}^i) + \\
&\qquad + 2q^1\{T_i, T_j\}q^{1\dagger} z^i \bar{z}^j + 2q^{2\dagger}\{T_i, T_j\}q^2 z^i \bar{z}^j\Big)\,. \qquad (4.27)
\end{aligned}
$$

Here we have split the $\mathcal{N}=2$ hypermultiplets into two $\mathcal{N}=1$ chiral multiplets $(q^1, q^{2\dagger})$, thus if q^1 transforms in the fundamental representation of $\mathrm{U}(N)$ q^2 transforms in the anti-fundamental representation of $\mathrm{U}(N)$. It is clear from the structure of the terms in the scalar potential in (4.27), this transformation property of the hypermultiplets is obeyed.

Using the fact that we have $\mathcal{N}=1$ symmetry and the information from the full scalar potential, we can write down the action for the spontaneously broken theory in a manifestly $\mathcal{N}=1$ invariant way, i.e. in superspace. For a theory with a polynomial superpotential for z we choose the following prepotential

$$
\mathcal{F} = \left(-\frac{e}{m} + i\left|\frac{\xi}{m}\right|\right)\mathrm{Tr}\,\frac{(z^2)}{2} + z^m\sum_{k=1}^{p}\frac{1}{k+1}g'_k\,\mathrm{Tr}(z^k)\,. \qquad (4.28)
$$

The prepotential in (4.28) satisfies the condition that at z^a, one has the required vacuum values as in (4.23). We can also consider adding a term independent of z^m in the above superpotential, but we will restrict ourselves the the form in (4.28). The superpotential is given by $\mathcal{W} = \frac{m}{2}\partial_m \mathcal{F}$. Now in addition to performing the similar scalings as in section 3, to restore the canonical dimensions of the scalars z and the superpotential we have to redefine $M \to M/\Lambda$. As in the previous section it will lead to a theory with a superpotential independent of Λ but with a non-trivial Λ dependent gauge kinetic term and a Kähler term for the scalars. Using the scalar potential in (4.27) we can summarize the resulting theory in $\mathcal{N}=1$ superspace language if we set the background $\mathcal{N}=2$ supergravity fields to zero for convenience. The $\mathcal{N}=2$ vector multiplet breaks up into a $\mathcal{N}=1$ gauge multiplet W and a massive $\mathcal{N}=1$ chiral multiplet Φ in the adjoint representation of $\mathrm{U}(N)$. The $\mathcal{N}=2$ hypermultiplets break up into two $\mathcal{N}=1$ chiral multiplets Q_1 and Q_2 in the fundamental and anti-fundamental representation of $\mathrm{U}(N)$. The $\mathcal{N}=1$ superspace effective action, obtained from spontaneously broken $\mathcal{N}=2$ action, is given by

$$\begin{aligned} S &= S_{\text{Potential}} + S_{\text{Kahler}} + S_{\text{Gauge}} + S_{\text{Hyper}}\,, \\ S_{\text{Potential}} &= \int d^2\theta \left(\mathrm{Tr}(Q_1\Phi Q_2 + \frac{m}{2}\mathcal{W}(\Phi) + \mathrm{Tr}(Q_1 M Q_2)\right), \\ S_{\text{Gauge}} &= \int d^2\theta \tau_{ab}(\Phi, \Phi,^m) W^a W^b\,, \\ S_{\text{Kahler}} &= \int d^4\theta \mathcal{K}(\Phi, \Phi_m; \Phi^\dagger, \Phi^\dagger_m) \\ S_{\text{Hyper}} &= \int d^4\theta (Q_1^\dagger Q_1 + Q_2^\dagger Q_2)\,. \end{aligned} \tag{4.29}$$

Note that a, b run over $1, \ldots, N+1$, which include the U(1) gauge multiplet under which the chiral multiplets Q_1 and Q_2, in the $\bar{\mathbf{N}}$, $\mathbf{N}$ of $\mathrm{U}(N)$ respectively, are not charged. We have also explicitly indicated the dependence of the gauge function τ and the Kähler kinetic term on the chiral multiplet corresponding to z^m, the Kähler metric is given by τ_{ab}. The kinetic term for the hypermultiplets is the standard one as in the $m_p \to \infty$ limit the metric on the quaternionic space is flat (4.6). It is clear that the superpotential does not depend on Φ_m for the choice of the prepotential given in (4.28). Note that in the superpotential chiral multiplets coming from the hypers couple to those from the vectors only linearly. This is a consequence of the fact that this theory is obtained from $\mathcal{N}=2$ theory.

Although in this section we have focused on obtaining chiral multiplets in the fundamental and anti-fundamental representation of $\mathrm{U}(N)$ gauge group, the generalization to arbitrary (complex) representation is straightforward.[2] Indeed, for an R-dimensional representation of $\mathrm{U}(N)$, we can start with the quaternionic manifold $\mathrm{U}(2,1+R)/\,\mathrm{U}(2)\times\mathrm{U}(1+R)$ for the hypermultiplets. The gauging now is done exactly as above with the representation matrix $T^i_{N\times N}$ in the eq. (4.18) for g_i replaced by representation matrices $T^i_{R\times R}$. Since the representation is unitary it is clear that $T^i_{R\times R}$ are in the Lie algebra of $\mathrm{U}(R)$. Furthermore, if the representation R is reducible, then we can include different masses for different irreducible factors in the equation (4.17) for g_0. The rest of the analysis goes exactly as above leading upto the $\mathcal{N}=1$ action (4.29), where now Φ appearing in the term $Q_1\Phi Q_2$ is in the R-dimensional representation.

[2]The resulting models are always vector-like, i.e. if a chiral multiplet transforms in a complex representation then there is another chiral multiplet in the complex conjugate representation.

Generalization to other gauge groups is also straightforward. For a gauge group G one starts with a symplectic section (X^Λ, F_Λ) exactly as given in (4.1) with $n = \dim(G)$. The Kähler potential and the Killing vectors are the same as in eqs. (2.4) and (2.5), except that the structure group is G and the indices i, j, k run from 1 to $\dim(G)$. One can also get matter in arbitrary unitary representation R of group G by starting with the quaternionic manifold given in the previous paragraph and replacing T^i's by the corresponding representation of G. In particular we can get quiver theories by choosing G to be the product of $\prod_{a=1}^{k} \mathrm{U}(N_a)$ modulo the center of mass U(1) (the role of the center of mass U(1) is played here by one of the U(1)'s which gauges a translational symmetry under which the remaining hypers are neutral). We can get the bi-fundamental chiral fields by gauging a suitable quaternionic manifold. For example, for A_k quiver theory, we can choose the quaternionic manifold $\mathrm{U}(2, n+1)/\,\mathrm{U}(2) \times \mathrm{U}(n+1)$ with $n = \sum_{a=1}^{k-1} N_a N_{a+1}$ and gauge in an obvious way by means of suitable representation matrices T^i. In fact, the example discussed in detail in this section, is an example of A_2 quiver theory with the gauge group $\mathrm{U}(N) \times \mathrm{U}(1)$, where the center of mass U(1) is the extra U(1) that gauges one of the translational symmetries of the hypers.

5. Chiral ring relations from $\mathcal{N} = 2$

In this section we derive the chiral ring relations from the solutions of the Bianchi identities of $\mathcal{N} = 2$ supergravity. We look for the appropriate solutions of the Bianchi identities from [11] and expand them around the vacuum which partially breaks supersymmetry, retaining the relevant $\mathcal{N} = 2$ gravity backgrounds. Our discussion will be focused on the case of $\mathcal{N} = 2$ gravity coupled to $\mathrm{U}(N)$ vector multiplets along with the minimal hypermultiplet responsible for partial breaking of supersymmetry, this was the case discussed in section section 3. We will briefly indicate the derivation of the chiral ring relations for matter in the fundamental representation.

Before we go into the details, we will first indicate the strategy followed in deriving the chiral the ring relations. Consider a $\mathcal{N} = 2$ chiral superfield $V^{(M_1, M_2, \ldots)}$, here the symbols $(M_1, M_2, \ldots)$ refer to the either internal symmetries or to bosonic and spinorial indices in superspace. We obtain relations in the chiral ring of $\mathcal{N} = 1$ supersymmetry by considering the following $\bar{D}$ exact quantity

$$\bar{D}^{1\dot{\alpha}}(D_{\alpha\dot{\alpha}} V^{(M_1, M_2, \ldots)}) = [\bar{D}^{1\dot{\alpha}}, D_{\alpha\dot{\alpha}}] V^{(M_1, M_2, \ldots)} . \tag{5.1}$$

Here the 1 in superscript of the covariant derivative refers to the unbroken $\mathcal{N} = 1$ supersymmetry. To write down the above equality we have used the fact that the superfield V is chiral, thus the commutator of the covariant derivatives in (5.1) can be set to zero in the $\mathcal{N} = 1$ chiral ring. From the definition of covariant derivatives in superspace we have the following

$$\begin{aligned}(\mathcal{D}_N \mathcal{D}_O \; - \; &(-1)^{no} \mathcal{D}_O \mathcal{D}_N) V^{(M_1 M_2 \ldots)} \\ &= -R_{NOP}^{M_1} V^{(P M_2 \ldots)} - R_{NOP}^{M_2} V^{(M_1 P \ldots)} - \cdots - T_{NO}^{P} \mathcal{D}_P V^{(M_1 M_2 \cdots)} .\end{aligned} \tag{5.2}$$

Thus the above combination of torsions, curvatures acting on a chiral superfield vanishes in the chiral ring, providing the relation we are looking for. To make our job simpler we will actually look at the commutator of covariant derivatives acting on the bottom component of a $\mathcal{N}=1$ superfield and then promote the identity we obtain to a $\mathcal{N}=1$ superfield relation. To evaluate the torsion and the curvatures involved in identities like (5.2) we appeal to the solutions of the Bianchi identities found in [11]. Consider the following Bianchi identity

$$\nabla^2 V^M = V^N R_N^M \,, \tag{5.3}$$

here V^M is a superfield valued form in $\mathcal{N}=2$ superspace, ∇ is the covariant derivative. Given a set of vielbein's in $\mathcal{N}=2$ superspace we can expand the covariant derivative as $\nabla = E^M \mathcal{D}_M$, where E^M are the supervielbein's in $N=2$ superspace. For example the $\theta=0$ components of the supervielbein in $\mathcal{N}=2$ superspace can be chosen to be $E^M = (V_\mu^a dx^\mu, \psi_\mu^A dx^\mu, \psi_{\mu A} dx^\mu)$.[3] In (5.3) the curvature R_N^M is a two form valued in superspace. Expanding the left hand side in flat indices we get

$$\begin{aligned} V^N R_N^M &= \nabla(E^N \mathcal{D}_N V^M)\,, \\ &= E^N \nabla(\mathcal{D}_N V^A) + \nabla(E^N)\mathcal{D}_N V^M\,, \\ &= E^N E^O \mathcal{D}_O \mathcal{D}_N V^M + T^N \mathcal{D}_N V^O\,. \end{aligned} \tag{5.4}$$

Note that in the last equality in the above equation one has the torsion piece and the commutator of the covariant derivatives. We can obtain the commutator relevant to (5.1) by looking for specific components in the above equation, namely we look for components with one spatial index, the $\alpha\dot\alpha$ component and one antichiral index in the $\mathcal{N}=1$ superspace. The equations (A.6) to (A.18) along with (A.26), (A.27) and (7.58) of [11] provide the information of the curvatures in (5.4), while the torsions can be read from equations (A.23) and (A.24) and (A.25).

We will now show how this procedure works in detail. Consider the following Bianchi identity obtained in [11]

$$\nabla^2 z^i = g\left(F^\Lambda - \bar L^\Lambda \bar\psi_A \wedge \psi_B \epsilon^{AB} - \bar L^\Lambda \bar\psi^A \wedge \psi^B\right) k_\Lambda^i(z) \tag{5.5}$$

where z^i is thought of as zero form in $\mathcal{N}=2$ superspace. Expanding the left hand side in terms of flat space derivatives we obtain

$$\nabla^2 z^i = E^M E^N \mathcal{D}_N \mathcal{D}_M z^i + T^M \mathcal{D}_M z^i\,. \tag{5.6}$$

The first term in the right hand side of the above equation is the commutator of the covariant derivatives. We need to look for indices with either M or N being the equal to the one spatial index and one anti-chiral spinorial index in the $\mathcal{N}=1$ superspace, to obtain the commutator of covariant derivative in (5.1). Now consider the torsion term in (5.6), from [11] equation (A.23), we see that a torsion with bosonic component vanishes,

[3] We have followed the notation of [11] to indicate the chiralities of the spinors. For the gravitinos, ψ^A is antichiral and ψ_A is chiral.

therefore M must take values along the spinorial directions, this allows us to write the torsion contribution as

$$T^M \mathcal{D}_M z^i = \rho_A \mathcal{D}^A z^i + \rho^A \mathcal{D}_A z^i \,. \tag{5.7}$$

From the equation

$$\nabla z^i = \bar{Z}^i_a V^a + \bar{\lambda}^{iA} \psi_A \tag{5.8}$$

we see that the second term of (5.7) drops out, as the covariant derivative of z^i does not have any component in the anti-chiral directions. The component of the torsion relevant for us is

$$\rho_A = \left[igS_{AB}\eta_{ab} + \epsilon_{AB}(T^-_{ab} + U^+_{ab}) \right] \gamma^b \psi^B \wedge V^a + \cdots \,, \tag{5.9}$$

Now we look for the components which contribute from the curvature on the right hand side of (5.5), again the component relevant for us is

$$F^\Lambda = \left(i f^\Lambda_i \bar{\lambda}^{iA} \gamma_a \psi^B \epsilon_{AB} \right) \wedge V^a + \cdots \,. \tag{5.10}$$

Equating the coefficients with one anti-chiral spinor index and a spatial index of (5.5) and (5.6), we obtain

$$ig f^\Lambda_j \bar{\lambda}^{jA} \gamma_a \epsilon_{AB} k^i_\Lambda = \bar{\lambda}^{iA} \left(igS_{AB}\eta_{ab} + \epsilon_{AB}(T^-_{ab} + U^+_{ab})\gamma^b \right) + [\mathcal{D}_B, \mathcal{D}_a] z^i \,. \tag{5.11}$$

To extract the commutator in (5.1) we need to consider the the anti-chiral spinor index to be that which corresponds to the unbroken $\mathcal{N} = 1$. From the breaking pattern discussed in section 3, this supersymmetry corresponds to $B = 1$. Substituting the fields, including the $\mathcal{N} = 2$ gravity backgrounds and the scalings involved in obtaining the rigid limit, we obtain

$$T^-_{ab} \bar{\lambda}^{i2} = i \bar{\lambda}^{j2} \gamma_a f^i_{jl} z^l \,. \tag{5.12}$$

As we have seen from the calculation of the mass of the fermions in the spontaneously broken theory in (3.10), the gauginos of the $\mathcal{N} = 1$ theory correspond to λ^{i1} which we denote by λ^i. Thus the above equation reduces, after a conjugation and some gamma matrix manipulation to

$$i f^i_{jk} \lambda^j_\alpha z^k = \frac{1}{2} T^-_{\alpha\beta} \lambda^{\beta i} \tag{5.13}$$

or in matrix notation to

$$[\lambda_\alpha, z] = \frac{1}{2} T_{\alpha\beta} \lambda^\beta \tag{5.14}$$

where[4] $T_{\alpha\beta} = 2\gamma^{ab}_{\alpha\beta} T^-_{ab}$ and T^- stands for the self-dual components of the graviphoton field strength background, we have suppressed the superscript $^{(b)}$ denoting the background for convenience.

There are two methods to obtain the second chiral ring relation. We will present the quick method here to have an understanding of the structure of the second ring relation and then present the detailed method using the Bianchi identities. Thinking of the relation

[4]The factor 2 occurs in passing from the conventions of the Lorentz generators of [11] to that of [23]. We use the conventions of the latter to write the chiral ring relations in their final form.

in (5.14) as a relation in $\mathcal{N}=2$ superspace, we can obtain the second ring relation by performing a supersymmetry transformation with the broken supersymmetry generators, that is generators with the 2 index. Thus the second ring relation is given by

$$if^i_{jk}\lambda^j_\alpha\lambda^k_\beta = \frac{1}{2}T_{\alpha\beta}D\delta^{in} + \frac{1}{2}\psi_{\alpha\beta\gamma}\lambda^{\gamma i} \tag{5.15}$$

where $D\delta^{in}$ is the expectation value of the D-term that we have turned on due to gauging from the hypermultiplet.[5] We will see that in the detailed analysis $D = 2\Lambda^2 m$ and $\psi_{\alpha\beta\gamma}$ is the $\mathcal{N}=1$ gravitino field strength background. To obtain (5.15) from (5.13) we we have used the following supersymmetry transformations

$$\delta_2 z^i = \lambda^i_\alpha\eta^\alpha\,, \qquad \delta_2\lambda^{\alpha i} = D\delta^{in}\eta^\alpha\,, \qquad \delta_2 T_{\alpha\beta} = -\psi_{\alpha\beta\gamma}\eta^\gamma. \tag{5.16}$$

Let us now see in detail how the second ring relation comes about using Bianchi identities. Consider the Bianchi identity which involves ∇^2 acting on λ^{iA}

$$\nabla^2\lambda^{iA} + \frac{1}{4}\gamma_{ab}R^{ab}\lambda^{iA} + \frac{i}{2}\hat{K}\lambda^{iA} + \hat{R}^i_j\lambda^{jA} - \frac{i}{2}\hat{R}^A_B\wedge\lambda^{iB} = 0\,. \tag{5.17}$$

Again following the similar procedure of expanding the left hand side of the above equation in flat space indices and looking for the components with one anti-chiral spinorial index and one spatial index, we obtain the following equation

$$\begin{aligned}\frac{-i}{4}\eta_{ca}\left(2\gamma^{[d}\rho_B^{b]c} + \gamma^c\rho_B^{db}\right)(\gamma_{db}\lambda^{iA}) - i\epsilon_{CB}\gamma_a\lambda^{kC}f^i_{jk}\lambda^{jA} + \\ +\sqrt{2}\epsilon_{CB}T^-_{ab}\gamma^b W^{iAC}m_p\Lambda + [D_B, D_a]\lambda^{iA} = 0\,.\end{aligned} \tag{5.18}$$

Here we have substituted for all the fields and their scalings involved, except W^{iAC}. Now consider the component with $B=1$ and $A=2$. After some gamma matrix manipulation and using the on shell conditions on the gravitino background we obtain the following chiral ring relation.

$$iT_{\alpha\beta}\Lambda^2 m\delta^{in} + \frac{1}{2}\psi_{\alpha\beta\gamma}\lambda^{\gamma i} = if^i_{jk}\lambda^j_\alpha\lambda^k_\beta\,. \tag{5.19}$$

In the above equation we have substituted the value of W^{iAC} at the $\mathcal{N}=1$ vacuum, we will show subsequently how fluctuations around this vacuum value are $\bar{D}$ exact terms. The gravitino field strength $i\rho_1^{ab}$ has been replaced by ψ^{ab}.[6]

It is important to note that the for the component $i=n$, the U(1) part the right hand side of (5.19) vanishes and we are left with the relation

$$iT_{\alpha\beta}\Lambda^2 m + \frac{1}{2}\psi_{\alpha\beta\gamma}\lambda^{\gamma n} = 0\,. \tag{5.20}$$

This is in contrast with the proposal of [4] and [5], there it was assumed that the presence of the graviphoton renders the gaugino non-Grassmanian, thus according to this proposal there would be a contribution from the the abelian part on the right hand side of (5.19).

[5] It is the 22 component in the supersymmetry variation (3.5).

[6] This factor of i occurs when changing from the signature conventions of [11] to [23].

We see here that within the framework of $\mathcal{N}=1$ theories obtained from spontaneously broken $\mathcal{N}=2$ theories, the presence of the graviphoton in the chiral ring relations is a consequence of Bianchi identities and traditional supergravity tensor calculus. As a result there is no drastic change in the Grassmann nature of the gaugino superfield or of the superspace coordinates.

As the two chiral ring relations in (5.13) and (5.19) are written in terms of bottom components of $\mathcal{N}=1$ superfields, we can promote these equations to an equation in $\mathcal{N}=1$ superspace. Thus the chiral ring relations are given by

$$\begin{aligned} [W_\alpha, \Phi] &= -\frac{i}{2\Lambda^2 \tilde{m}} F_{\alpha\beta} W^\beta \,, \\ [W_\alpha, W_\beta] &= F_{\alpha\beta} + 2G_{\alpha\beta\gamma} W^\gamma \,. \end{aligned} \tag{5.21}$$

Here we have used the fact that the bottom component of $G_{\alpha\beta\gamma}$ is given by $\psi_{\alpha\beta\gamma}/4$ [24]. We have also rescaled the graviphoton with $F_{\alpha\beta} = m\Lambda^2/\sqrt{N} T_{\alpha\beta}$. This ensures that the dimensions for the graviphoton is 3. The factor of $\sqrt{N}$ appears in writing the (5.19) as a matrix equation since the U(1) generator is $I/\sqrt{N}$, finally $m = \sqrt{N}\tilde{m}$.

In deriving the ring equations (5.21) from the Bianchi identities, we have neglected the fluctuations in the D terms and just substituted the background U(1) valued D-term proportional to $\Lambda^2 m$. We argue that the fluctuations to the D-terms arising from the expansion around the $\mathcal{N}=1$ vacuum are $\bar{D}$ exact terms and thus vanish in the chiral ring. Indeed $\mathcal{N}=2$ vector multiplet can be organized into a constrained chrial $\mathcal{N}=2$ superfield as follows

$$\Psi = \Phi + \theta^2 W + (\theta^2)^2 D \,. \tag{5.22}$$

Here the auxiliary D is an $\mathcal{N}=1$ superfield whose bottom component gets an expectation value at the vacuum. The reality constraint on the $\mathcal{N}=2$ superfield is given by

$$(\epsilon^{AB} D_A \sigma^{\mu\nu} D_B)^2 \Psi = -96\Box\Psi^\dagger \,. \tag{5.23}$$

Acting by $(\bar{D}_1)^2$ on both sides of the equation we obtain

$$\begin{aligned} 4\Box(D_2)^2\Psi &= 3(\bar{D}_1)^2\Box\Psi^\dagger \,, \\ 4\Box D &= 3(\bar{D}_1)^2\Box\Phi^\dagger \,. \end{aligned} \tag{5.24}$$

Thus D is $\bar{D}$ exact, but a constant expectation value of D is not subject to this constraint, due to the presence of the d'Alembertian on both sides of the equation. Thus fluctuations in the D term can be neglected in the second chiral ring equation of (5.21).

The chiral ring relations for the hypermultiplets do not receive any gravitational corrections. This can be seen by careful considerations of the Bianchi identities involving the hypermultiplets from [11] and the scalings given in (4.21), but here we present a quick argument. Consider the commutator in (5.1) acting on the chiral multiplet Q_1 obtained from the hypermultiplet, there is only one contribution to the curvature term, due to the presence of the gauge index on Q_1. As Q_1 is a scalar it does not receive any contribution from the gravitino field strength because the chiral field in (5.1) must carry at least one spinor index to receive a contribution. From the previous examples we have seen that the torsion term in (5.1) always comes with a covariant derivative in the 2 direction, as a $\mathcal{N}=2$ superfield. If D_2Q_1 is non zero, it should be proportional to Q_2 as D_2 mixes the two chiral multiplets of hypers. But D_2Q_1 cannot contain any chiral components since Q_2 is in the anti-fundamental representation, while D_2Q_1 must be in the fundamental representation. Therefore, from this analysis we see that the only contribution to the commutator in (5.1) is from the curvature term and that is because of the gauge index on Q_1. Thus the chiral ring relations for the hypermultiplets are given by

$$Q_1 W_\alpha = 0\,, \qquad W_\alpha Q_2 = 0\,. \tag{5.25}$$

This completes our analysis of the gravitational deformed chiral rings using the partially broken $\mathcal{N}=2$ theory.

6. Generalized Konishi anomaly equations

In ref. [8], we had obtained gravitational corrections to the effective superpotential using generalized Konishi anomaly equations. In that derivation we had assumed that the $\mathcal{N}=1$ gauge function and Kähler functions at the classical level were trivial and only the classical superpotential was an arbitrary single trace polynomial in the chiral field. Furthermore the chiral ring relation that was used, although similar to eqs.(5.21), was not exactly the same. Specifically the ring relations used there were

$$[W_\alpha, \Phi] = 0 \tag{6.1}$$

$$[W_\alpha, W_\beta] = F_{\alpha\beta} + 2G_{\alpha\beta\gamma}W^\gamma\,. \tag{6.2}$$

So while the second ring relation is the same as in eq.(5.21), the first one differs. Note that the right hand side of the first equation (5.21), scales as $1/\Lambda^2$. Therefore in the limit of the supersymmetry breaking scale $\Lambda \to \infty$, the ring relations derived here reduce to the ones of ref. [8]. Moreover, in this limit, the gauge function (3.17) and Kähler metric become field independent and therefore in this limit we reproduce the situation considered in ref. [8].

In this section we will analyze the modifications in the anomaly equations when Λ is finite and show that there are Λ-dependent corrections. We will also see that the consistency of the anomaly equations in the presence of graviphoton field strength crucially requires the $\mathcal{N}=2$ relation between the superpotential and the gauge function, namely that they come from the prepotential.

Since W_α and Φ do not commute in the chiral ring (5.21), at first sight it appears that the derivation of generalized Konishi anomaly equations become very difficult as now the ordering of Φ with respect to W_α inside traces cannot be ignored. However by a field redefinition ring relations of (5.21) can be transformed into eq. (6.2). Indeed from the second ring relation we can obtain the following relation

$$[W_\alpha, W^2] = -2F_{\alpha\beta}W^\beta\,. \tag{6.3}$$

Using this fact it is easy to see that by a field redefinition

$$\Phi \to \Phi + i\frac{1}{4\tilde{m}\Lambda^2}W^2 \tag{6.4}$$

the first ring relation of eq (5.21) goes over to that of eq. (6.2). By substituting the above field redefinition in the action we can read off the gauge function and the superpotential in terms of the redefined chiral field Φ. In appendix B, it is shown that the superpotential remains unchanged while the gauge function modulo $\bar{D}$ exact terms is given by,

$$\begin{aligned} S_{\rm Gauge} &= -\frac{i}{4}\int d^2\theta\, {\rm Tr}(W^2) + S_{\rm Gravity}\,, \\ S_{\rm Gravity} &= \frac{\alpha^2}{6}\int d^2\theta\, {\rm Tr}[\mathcal{W}''(\Phi)(G^2W^2 + (G\cdot F)^\alpha W_\alpha)]\,, \qquad \alpha = \frac{i}{4\tilde{m}\Lambda^2}\,. \end{aligned} \tag{6.5}$$

Thus the Λ depdendence in entirely contained in the gravity induced gauge kinetic term $S_{\rm Gravity}$, while in the *b*are gauge kinetic term $-\frac{i}{4}(W^2)$ the Λ dependence drops out. As this point is important we will demonstrate this simplification for the first Λ dependent term here. This Λ dependent term arises from the cubic term of the prepotential in (3.14). We see from (3.17) the contribution of the cubic term to the kinetic term for the gauginos is

$$-\frac{g_1}{\Lambda^2}\frac{i}{4}\,{\rm Tr}(\Phi W^2)\,. \tag{6.6}$$

Now performing the shift of (6.4) in the superpotential on the term $\tilde{m}\frac{g_1}{2}\Phi^2$one gets the same contribution as in (6.6) but with the opposite sign, thus this Λ dependent term cancels. It is important to stress here that this cancellation between the gauge kinetic terms coming from the original gauge function and the superpotential, crucially depends on the $N=2$ relation between the two. Had we taken an arbitrary gauge function, which is certainly allowed in $N=1$ theory, this cancellation would not have taken place (in terms of the redefined fields (6.4)) and there would have been a surviving non-trivial gauge function. We will show below that the resulting generalized Konishi anomaly equations then would have been inconsistent in the presence of graviphoton field strength.

The Kähler function, i.e. the kinetic term for the chiral fields, is however still a non-trivial function of redefined chiral fields for finite Λ. Note however that all these non-trivial Φ dependent terms are non-renormalizable. One might wonder if under a transformation $\Phi \to \Phi + f(\Phi)$ where f is an arbitrary function, needed to derive generalized Konishi anomaly equations, the non-trivial Kähler function would contribute. Since our equations are chiral ring equations (i.e. modulo $\bar{D}$ exact terms), the Kähler function can contribute only through generalized Konishi anomaly. We do not expect that the non-renormalizable terms would contribute to the anomaly, therefore the structure of anomaly contribution to the Schwinger-Dyson equations resulting from the above transformation would be governed by the trivial renormalizable (field independent) kinetic term.

It then follows that the equations derived in ref. [8] remain unchanged except for the Λ dependent contributions from the gravity induced gauge kinetic terms $S_{\rm Gravity}$. The effective superpotential receives a correction for finite Λ due to the presence of these terms which, this correction is evaluated in appendix C. However now we would like to show that if there had been a non-trivial bare gauge function (after the field redefinition (6.4)), we would have obtained inconsistent equations in the presence of graviphoton field strength. In the following Φ will always denote the redefined field (6.4) so that in the chiral ring it commutes with W_α. Recall that under an infinitesimal transformation $\delta\Phi = f(\Phi, W)$, the generalized Konishi anomaly is given by

$$\frac{1}{32\pi^2}\frac{\delta f_{ji}}{\delta\Phi_{k\ell}}A_{ij,k\ell}\,, \tag{6.7}$$

with

$$A_{ij,k\ell} = (W^2)_{kj}\delta_{i\ell} + \delta_{kj}(W^2)_{i\ell} - 2W^\alpha_{kj}W_{\alpha i\ell} + \frac{1}{3}G^2\delta_{kj}\delta_{i\ell}\,. \tag{6.8}$$

where the last term is the gravitational contribution to the anomaly. Now consider a transformation

$$\delta\Phi = f(\Phi, W) = F_{\alpha\beta}\frac{1}{z-\Phi} + 2G_{\alpha\beta\gamma}\frac{W^\gamma}{z-\Phi}\,. \tag{6.9}$$

The corresponding generalized Konishi anomaly vanishes (modulo D-terms). Indeed by using the anomaly (6.8) and the chiral ring relation (6.2) we find that the anomaly is proportional to:

$$2\,\mathrm{Tr}\left(\frac{W^2}{z-\Phi}\right)\left(F_{\alpha\beta}\,\mathrm{Tr}\left(\frac{1}{z-\Phi}\right) + 2G_{\alpha\beta\gamma}\,\mathrm{Tr}\left(\frac{W^\gamma}{z-\Phi}\right)\right) = 2\,\mathrm{Tr}(\frac{W^2}{z-\Phi})\,\mathrm{Tr}\left(\frac{\{W_\alpha W_\beta\}}{z-\Phi}\right) = 0 \tag{6.10}$$

where we have used the fact that in the chiral ring G^2 and $(F.G)_\gamma$ multiplied by F or G vanish (see eqs.(2.13)-(2.20) of ref. [8] for proof).

Under this transformation the change in classical superpotential vanishes:

$$\delta\mathcal{W}(\Phi) = F_{\alpha\beta}\,\mathrm{Tr}\left(\frac{\mathcal{W}'}{z-\Phi}\right) + 2G_{\alpha\beta\gamma}\,\mathrm{Tr}\left(\mathcal{W}'\frac{W^\gamma}{z-\Phi}\right) = \mathrm{Tr}\left(\mathcal{W}'\frac{\{W_\alpha W_\beta\}}{z-\Phi}\right) = 0\,. \tag{6.11}$$

Here and in the following prime denotes derivative with respect to Φ. The change in the gravity induced gauge kinetic term S_{Gravity} due to the transformation (6.9) also vanishes because in the chiral ring G^2 and $(G\cdot F)$ multiplied by either F or G vanishes. If there is a non-trivial bare gauge function $\mathrm{Tr}\,\tau(\Phi)W^2$ in the action, then the corresponding change is

$$\delta(\mathrm{Tr}\,\tau W^2) = F_{\alpha\beta}\,\mathrm{Tr}\left(\tau'\frac{W^2}{z-\Phi}\right) + 2G_{\alpha\beta\gamma}\,\mathrm{Tr}\left(\tau'\frac{W^\gamma W^2}{z-\Phi}\right) = F_{\alpha\beta}\,\mathrm{Tr}\left(\tau'\frac{W^2}{z-\Phi}\right) \tag{6.12}$$

where we have used the fact that

$$\mathrm{Tr}\left(\tau'\frac{W^\gamma W^2}{z-\Phi}\right) = \frac{1}{2}\,\mathrm{Tr}\left(\tau'\frac{\{W^\gamma, W^2\}}{z-\Phi}\right) \tag{6.13}$$

$$= -\frac{1}{3}\left(G^2\,\mathrm{Tr}\left(\tau'\frac{W^\gamma}{z-\Phi}\right) + (G.F)^\gamma\,\mathrm{Tr}\left(\tau'\frac{1}{z-\Phi}\right)\right) \tag{6.14}$$

and the fact that G^2 and $(G.F)^\gamma$ when multiplied by another G vanishes in chiral ring. In the second equality above, we have repeatedly made use of the second ring relation (6.2).

This shows that the change in the gauge kinetic term does not vanish if the bare gauge function τ is a non-trivial function of Φ (the only exception for single trace gauge function is if τ' is proportional to G^2). Thus for a non-trivial bare gauge function the resulting Schwinger-Dyson equations would be inconsistent with the chiral ring relations (6.2). As shown in the appendix B, the triviality of the bare gauge function is a consequence of the precise $\mathcal{N}=2$ relation between the superpotential and the gauge function, which arises due to the fact that the $\mathcal{N}=1$ theory is obtained by partial spontaneous breaking of $\mathcal{N}=2$ theory.

The superpotential does receive corrections because of the presence of the gravity induced gauge kinetic term, this can be read off from the corrections to the gauge invariant operator $T(z) = \mathrm{Tr}\,\frac{1}{z-\Phi}$ evaluated in (C.11) of appendix C.

7. Conclusions

In this paper we have discussed the $\mathcal{N}=1$ effective field theory arising in the rigid limit of a partially broken $\mathcal{N}=2$ supergravity coupled to vector- and hyper-multiplets, in the presence of a non-trivial background graviphoton $F_{\alpha\beta}$ and gravitational superfield $G_{\alpha\beta\gamma}$. We have shown that the effective field theory one obtains is the one discussed in [1] and subsequent literature. Moreover, we have also derived the chiral ring relations in the presence of the above backgrounds. We have shown that they are precisely those used in the literature [4, 5, 24, 8] to connect the generalized gravitational superpotentials of the SYM theory to the coefficients of the topological expansion in the corresponding matrix model integral. We should stress however that in our approach the chiral ring is a consequence of the Bianchi identities of the underlying $\mathcal{N}=2$ theory and therefore no drastic change in the Grassmann nature of the gaugino superfield or of the superspace coordinates is postulated.

Another important result of our analysis, supporting the $\mathcal{N}=2$ interpretation, is that, in the presence of $F_{\alpha\beta}$, consistency of the loop equations requires that a non trivial gauge function, in principle arbitrary for an $\mathcal{N}=1$ SYM theory, should in fact be related to the superpotential precisely in the way given by $\mathcal{N}=2$ supersymmetry.

A point which deserves better understanding is whether our effective field theory can be derived from the effective action of wrapped D5-branes. Whereas the relation of fluxes to gauging seems to be well understood on the closed string side of the closed/open string duality, on the D5-branes side this is not completely clear. We recall that the crucial ingredient for this partial breaking is the gauging of two translational symmetries in the hypermultiplet space; with the two U(1) gauge fields being the graviphoton and another gauge field that decouple completely in the rigid limit. Here we indicate possible mechanisms for this gauging on the D-brane side. The obvious candidate for the latter is the center of mass U(1) of the D-brane system. The hypermultiplets that can naturally play a role in this problem are the universal one (associated to the dilaton) and the one associated with the (1,1) form dual to the 2-cycle which is wrapped by the D-5 brane. Consider the Chern-Simons term on the D-brane world-volume

$$S_{cs} \sim \int d^6 x C_4 \wedge \mathrm{Tr}\, F \tag{7.1}$$

where F is the world-volume gauge field strength and C_4 is the RR 4-form potential. Writing $C_4 = T_2 \wedge \omega_{(1,1)}$ where $\omega_{(1,1)}$ is the (1,1)-form dual to the 2-cycle wrapped by the D-brane and T_2 is a 2-form in the remaining directions, and dualizing T_2 in the non-compact 4-dimensional part of the world volume, we find a coupling of the form $\mathrm{Tr}\, A^\mu \partial_\mu b$

where b is the scalar field dual to T_2. Thus we see that the center of mass U(1) gauges the translational symmetry of the scalar field b. Note also that writing the gauge field as $A = A_{cm}\mathbf{1}/\sqrt{N} + A^i T_i$ where T_i are the SU(N) generators and the factor of $\sqrt{N}$ is included so that the kinetic term for A_{cm} is normalized to unity, we find that the Chern-Simons coupling term is $\sqrt{N} A^\mu_{cm}\partial_\mu b$. This is in accordance with the scaling of charge m as $\sqrt{N}\tilde{m}$ observed in the comments following eq. (5.21).

The issue of the second U(1) gauging by graviphoton is less clear in the context of D-branes. In IIB theory under consideration, the graviphoton appears from the RR 5-form field strength $F_5 = F_2^{gr} \wedge \Omega$ where Ω is the holomorphic 3-form on the (non-compact) Calabi-Yau space. Given the fact that in the presence of D5- branes there is a non-zero flux of the associated RR field strength F_3 through the non-compact Calabi-Yau space, the 10-dimensional Chern-Simons term $\int d^{10}x F_5 \wedge F_3 \wedge B_2$ could in principle gauge the translational symmetry of the scalar field dual to the NS-NS 2-form potential B_2. The charge then would be proportional to the integral $\int_{CY} \Omega \wedge F_3$ over the Calabi-Yau space. Thinking of the non-compact Calabi-Yau as a K_3 fibered over a plane, the holomorphic 3-form can be written as $\Omega = \omega \wedge dz$, where ω is the holomorphic 2-form inside K_3 and dz is the holomorphic 1-form on the complex plane, thus $\Omega = d\alpha$ where

$$\alpha = \int_{z=z_0}^{z} \omega \wedge dz \,, \tag{7.2}$$

therefore

$$\int_{CY} \Omega \wedge F_3 = \int \alpha \wedge dF_3 + \int \alpha \wedge F_3|_\infty \tag{7.3}$$

here by ∞ we mean the boundary on the complex plane at infinity. The first term in the above equation is finite, infact dF_3 is localized at the position of the brane. However one needs to put a cut off to give a meaning to the second term. Thus it is not clear if the theory on D5-brane wrapped on a 2 cycle of a non-compact Calabi-Yau provides a realization of the spontaneously broken theory considered in this paper. It will be interesting to study this issue further.

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A. Conventions

We list here the conventions for the flat hyper Kähler manifold. The manifold is parametrized by the coordinates $q = (q^{Au}, \bar{q}^{A\bar{u}})$, where $A = 1, 2$ and $u = 1, \ldots, m$. They form a set of $2m$ complex coordinates. The metric on this manifold is given by

$$ds^2 = dq^{Au} dq^{B\bar{u}} \delta_{AB}\delta_{u\bar{u}} \,. \tag{A.1}$$

We will denote this metric as $h_{\rm uv}$, here u, v refer to the entire set $2m$ holomorphic and the anti-holomorphic coordinates, thus $h_{AuB\bar{u}} = \frac{1}{2}\delta_{AB}\delta_{u\bar{u}}$. The vielbein one form is given by

$$\mathcal{U}^{Aai} = \mathcal{U}^{Aai}_{Bu} dq^{Bu} + \mathcal{U}^{Aai}_{B\bar{u}} d\bar{q}^{B\bar{u}} \,. \tag{A.2}$$

Here we have split the Sp$(2m)$ indices as ai with $a = 1, 2$ and $i = 1, \ldots, m$. The explicit form for the vielbein are

$$\mathcal{U}_{Bu}^{Aai} = \frac{1}{\sqrt{2}}\delta^{a1}\epsilon^{AB}\delta^{iu}\,, \qquad \mathcal{U}_{B\bar{u}}^{Aai} = \frac{1}{\sqrt{2}}\delta^{a2}\delta^{AB}\delta^{iu}\,. \tag{A.3}$$

The vielbein defined this way satisfy the following required conditions for the hyper-Kähler manifold

$$\begin{aligned} h_{\mathrm{uv}} &= \mathcal{U}_{\mathrm{u}}^{Cai}\mathcal{U}_{\mathrm{v}}^{Dbj}\epsilon_{CD}\epsilon_{ab}\delta ij\,, \\ (\mathcal{U}^{Aai})^* &= \epsilon_{AB}\epsilon_{ab}\delta_{ij}\mathcal{U}^{Bbj}\,, \\ h_{\mathrm{uv}}\epsilon^{AB} &= (\mathcal{U}_{\mathrm{u}}^{Aai}\mathcal{U}_{\mathrm{v}}^{Bbi} + \mathcal{U}_{\mathrm{v}}^{Aai}\mathcal{U}_{\mathrm{u}}^{Bbi})\epsilon_{ab}\delta_{ij}\,. \end{aligned} \tag{A.4}$$

B. Λ independence of the bare gauge kinetic term

In this appendix we show that after the field redefinition (6.4) the Λ dependence of the gauge kinetic terms drops out modulo $\bar{D}$ terms for the $\mathcal{N} = 1$ theory obtained from partially broken $\mathcal{N} = 2$. We will need the following identities in the chiral ring which are obtained by simple algebraic manipulations from the basic ring relations.[7]

$$\begin{aligned} [W_\alpha, W^2] &= -2F_{\alpha\beta}W^\beta\,, \\ \{W_\alpha, W^2\} &= -\frac{2}{3}(G^2 W_\alpha + G_{\alpha\beta\gamma}F^{\beta\gamma})\,. \end{aligned} \tag{B.1}$$

To simplify the first ring relation in (5.21) we perform the following field redefinition

$$\Phi \to \Phi + \alpha W^2\,, \qquad \alpha = \frac{i}{4\tilde{m}\Lambda^2}\,. \tag{B.2}$$

The redefined Φ now commutes with W_α, but we have to perform the same shift in the action of the spontaneously broken theory. As we have seen in section 3. after breaking $\mathcal{N} = 2$ symmetry to $\mathcal{N} = 1$ we following obtain the superpotential

$$\mathcal{W} = \tilde{m}\sum_{k=1}^{n}\frac{g_k}{k+1}\,\mathrm{Tr}\,\Phi^{k+1}\,. \tag{B.3}$$

Apart from the trivial Λ dependent term in the gauge kinetic function this theory also has a non trivial gauge function obtained from the prepotential

$$\mathcal{F}' = \frac{1}{\Lambda^2}\sum_{k=1}^{n}\frac{g_k}{(k+1)(k+2)}\,\mathrm{Tr}\,\Psi^{k+2}\,. \tag{B.4}$$

A convenient way of writing the gauge function of the $\mathcal{N} = 1$ theory is $i\int d^2\hat{\theta}\mathcal{F}'(\Psi)$ and substituting $\Psi = \Phi + \hat{\theta}W_\alpha + \hat{\theta}^2 D$ in the prepotential and extracting out the coefficient of W^2.[8] Roughly this is given by the second derivative of the prepotential (B.4). However

[7] For a proof of these identities see [8].

[8] This definition of the gauge function gives the correct normalization for the usual kinetic term for the prepotential $\Psi^2/2$ which is $-i/4$

as Φ and W_α don't commute with each other we have to be careful of the orderings in higher powers of Ψ. Our aim now is to make the redefinition of (B.2) in both the gauge function and the superpotential and simplify the resulting higher powers of W^2 using the ring relations of (B.1). We claim that after the redefinition to the chiral multiplet which commutes with W^α the Λ dependence of the gauge kinetic term drops out.

As a first test we look at the terms which are of the type $\mathrm{Tr}(\Phi W^2)$. There are two contributions, one from the superpotential which is given by $\tilde{m}\alpha g_1 \,\mathrm{Tr}(\Phi W^2)$, the contribution from the gauge function is given by $-i\frac{1}{4\Lambda^2}\,\mathrm{Tr}(\Phi W^2)$ and thus for the value of α given in (B.2) they cancel. The $-\frac{1}{4}$ factor comes if one keeps track of the $\hat{\theta}$ orderings. Terms proportional to W^4 can be rewritten due to the following identity

$$\begin{aligned} \mathrm{Tr}(W^2W^2\Phi^k) &= \frac{1}{2}\,\mathrm{Tr}\{W^\alpha W_\alpha, W^2\Phi^k\}\,, \\ &= -\frac{1}{3}\,\mathrm{Tr}(G^2W^2\Phi^k + (G\cdot F)^\alpha W_\alpha \Phi^k)\,. \end{aligned} \tag{B.5}$$

We also note that

$$[W_\alpha, W^4] = 0\,. \tag{B.6}$$

From (B.5) and (B.6) it is easy to see that contribution proportional $(W^4)^p$ for $p > 1$ vanishes in the chiral ring. This is seen as follows

$$\begin{aligned} \mathrm{Tr}(W^4(W^4)^{p-1}\Phi^k) &= \frac{1}{2}\,\mathrm{Tr}(W^\alpha W_\alpha, W^2(W^4)^{p-1}\Phi^k)\,, \\ &= -\frac{1}{3}\,\mathrm{Tr}\left(G^2W^2 + (G\cdot F)^\alpha W_\alpha)(W^4)^{p-1}\Phi^k\right)\,. \end{aligned} \tag{B.7}$$

For dimensional reasons any reduction of powers of W in the last equation of (B.7) would involve higher powers of F or G. Any multiplication of F or G with G^2 or $G\cdot F$ vanishes in the chiral ring [8]. Therefore contributions proportional to $(W^4)^p$ vanishes for $p > 1$.

We are now left with terms proportional to $(W^2)^{2p+1}$. We will now show that they too vanish. Consider the contribution to $(W^2)^{2p+1}$ from terms proportional to g_k for sufficiently large k both from the superpotential and from the gauge kinetic term. The superpotential contributes the following term.

$$\mathcal{W}_{2p+1} = \tilde{m}\alpha^{2p+1}\frac{g_k}{k+1}\frac{(k+1)!}{(2p+1)!(k-2p)!}\,\mathrm{Tr}((W^2)^{2p+1}\Phi^{k-2p})\,. \tag{B.8}$$

We will denote the combinatorial factor in front by $N_s = {}^{k+1}C_{2p+1}$. From the gauge kinetic term the contribution to $(W^2)^{2p+1}$ can be organized into two kinds. One kind of terms can be manipulated due to the cyclic properties of the trace to a term proportional to $\mathrm{Tr}((W^2)^{2p+1}\Phi^{k-2p})$, the second kind of terms can be manipulated by the cyclic properties of the trace and (B.6) to a term proportional to $\mathrm{Tr}((W^2)^{2p-1}W^\alpha W^2 W_\alpha \Phi^{k-2p})$. We will now find the precise combinatorial coefficient in front of this term. The second type of terms arise from first choosing $k-2p$ Φ's out of $k+2$ terms and then looking for various arrangements of the W's such that there are odd number of W^2 between two single W^α's. This is because if there is an even number of W^2 between the two W_α they can be commuted through so that the two W_α become adjacent giving rise to W^2, and therefore gives a term of the first type. The number of ways of having odd number of W^2's between two W_α's is given by $p(p+1)$. This is argued as follows; from the remaining $2(p+1)$ Φ we need to take two W_α and the remaining $2p$ W^2. In doing so we must have odd number of W^2 in between the positions of the two W_α. To count the number of ways we can have odd number of W^2 between the two W_α, let n denote the position of the second W_α. Clearly $n = 3, \ldots, 2p+2$. Then the position of the first W_α must be $n - 2k$ for some integer k so that the number of W^2 in the middle is $2k - 1$. Clearly $k = 1, \ldots [\frac{n-1}{2}]$ where the square bracket means the integer part of the argument. The number of possible values of k therefore is $(n-1)/2$ for odd n and $(n-2)/2$ for even n. Writing $n = 2m+1$ for odd n and $2m+2$ for even n we see that the range of m in either case is from 1 to p and the number of possible k is m. Thus the total number of such terms is $2\sum_{m=1}^{p} m = p(p+1)$. All these terms, by using the cyclicity of the trace and by moving even powers of W^2 across W_α can be brought to the form $\mathrm{Tr}((W^2)^{2p-1} W^\alpha W^2 W_\alpha \Phi^{k-2p})$. Thus the total number of terms of the second type is $N_o = {}^{k+2}C_{k-2p}\, p(p+1)$, where the binomial factor comes from possible ways of picking $k - 2p$ Φ's out of $k+2$ superfields.

Let us now count the number of configurations of the first type namely, $\mathrm{Tr}((W^2)^{2p+1}\Phi^{k-2p})$. This is obtained by subtracting the total number of configurations in the gauge function proportional to $(W^2)^{2p+1}$ by N_o. The total number of such configurations is given by $N_t = {}^{k+2}C_2\, {}^kC_{2p}$.

Therefore the terms proportional to $g_k {W^2}^{2p+1}$ from both the superpotential and gauge function totally are given by

$$
\begin{aligned}
-\frac{i}{2\Lambda^2} g_k \frac{\alpha^{2p}}{(k+1)(k+2)} \Big((N_t - N_o)\, \mathrm{Tr}(W^2)^{2p+1}\Phi^{k-2p} + \\
+ N_o\, \mathrm{Tr}(W^2 W^\alpha W^2 W_\alpha (W^2)^{2(p-1)} \Phi^{k-2p}) \Big) + \\
+\tilde{m} \frac{g_k}{k+1} \alpha^{2p+1} N_s\, \mathrm{Tr}((W^2)^{2p+1}\Phi^{k-2p}) \,.
\end{aligned} \tag{B.9}
$$

Substituting the values of N_t, N_o, N_s and $\alpha = \frac{i}{4\Lambda^2 \tilde{m}}$, we see that the above term reduces to

$$
-\frac{2\alpha^{2p+1} N_0}{(k+1)(k+2)} \mathrm{Tr}(W^2 W^\alpha \{W_\alpha, W^2\} (W^2)^{2(p-1)} \Phi^{k-2p}) \,. \tag{B.10}
$$

This is because the combinatorial factors satisfy the relation

$$
N_0 = \frac{1}{4} \left(2N_t - (k+2) N_s\right) . \tag{B.11}
$$

Now from the second equation of (B.1) we see that the term in (B.10) is already proportional to G^2 or $G \cdot F$, therefore reducing the remaining W's by chiral ring identities ensures that such terms vanish.

What remains to compute is the term proportional to $(W^2)^2$ coming from the superpotential and the gauge function. From the superpotential this term is

$$
\tilde{m}\alpha^2 \frac{g_k}{k+1} {}^{k+1}C_2\, \mathrm{Tr}(\Phi^{k-1}(W^2)^2) \tag{B.12}
$$

while from the gauge function, since there is no difference between $\mathrm{Tr}(\Phi^{k-1}(W^2)^2)$ and $\mathrm{Tr}(\Phi^{k-1}W^\alpha W^2 W_\alpha)$ due to the cyclicity of trace, we obtain

$$-3\alpha\frac{i}{2\Lambda^2}\frac{g_k}{(k+1)(k+2)}\,{}^{k+2}C_3\,\mathrm{Tr}(\Phi^{k-1}(W^2)^2) \tag{B.13}$$

where the factor 3 in the front appears due the three possible arrangements of W^2 and the two single W_α's. Substituting the value of α we find that the two contributions (B.12) and (B.13) together give

$$-\,\tilde{m}\alpha^2 g_k\frac{k}{2}\,\mathrm{Tr}(\Phi^{k-1}(W^2)^2)\,. \tag{B.14}$$

We can write the entire contribution to the W^4 term compactly using (B.3) as

$$\begin{aligned} S_{\text{Gravity}} &= -\frac{\alpha^2}{2}\,\mathrm{Tr}[\mathcal{W}''(W^2)^2] \\ &= \frac{\alpha^2}{6}\,\mathrm{Tr}[\mathcal{W}''(G^2W^2+(G\cdot F)^\alpha W_\alpha]\,. \end{aligned} \tag{B.15}$$

In the second line of the above equation, we have used (B.5).

To conclude, we have shown here that after the field redefinition of Φ, so that the new field commutes with W_α, all the non-trivial Φ- dependent (and hence Λ-dependent) part of the bare gauge function cancels upto $\bar{D}$ exact terms. What remains is a gravity induced gauge function and a gravity induced term which is linear in W_α. Note that the terms in S_{Gravity} are all proportional to $1/\Lambda^4$, therefore they all vanish in the limit of the supersymmetry breaking scale $\Lambda\to\infty$. These additional terms in the effective action give rise to extra terms in the generalized Konishi anomaly equations considered [8].

C. $1/\Lambda$ corrections to the Konishi anomaly equations

In this section we derive the corrections to the Konishi anomaly equations due to the presence of the terms in (B.15) in the effective action. The gauge invariant operators constructed out of the basic fields of the $\mathcal{N}=1$ gauge multiplet and the chiral multiplet can be arranged into the following operators

$$\begin{aligned} \mathcal{R}(z)_{ij} &= -\frac{1}{32\pi^2}\left(\frac{W^2}{z-\Phi}\right)_{ij}, & R(z) &= \mathrm{Tr}\,\mathcal{R}(z)\,, \\ \rho_\alpha(z)_{ij} &= \frac{1}{4\pi}\left(\frac{W_\alpha}{z-\Phi}\right)_{ij}, & w_\alpha &= \mathrm{Tr}\,\rho_\alpha(z)\,, \\ \mathcal{T}(z)_{ij} &= \left(\frac{1}{z-\Phi}\right)_{ij}, & T(z) &= \mathrm{Tr}\,\mathcal{T}(z)\,. \end{aligned} \tag{C.1}$$

The three set of Ward indentities constraining the gauge invariant operators R, $w_\alpha(z)$ and $T(z)$ is obtained by considering an infinitesimal variation $\delta\Phi_{ij}=f_{ij}$ where f_{ij} is the matrix elements of the operators given in (C.1). This variation is anomalous and the anomaly in presence of gravity is given by

$$\frac{1}{32\pi^2}\frac{\delta f_{ji}}{\delta\Phi_{kl}}A_{ij,kl}\,, \tag{C.2}$$

with

$$A_{ij,kl} = (W^2)_{kj}\delta_{il} + \delta_{kj}(W^2)_{il} - 2W^{\alpha}_{kj}W_{\alpha il} + \frac{1}{3}G^2\delta_{kj}\delta_{il}\,. \tag{C.3}$$

The anomaly equations in presence of the gravity modified terms in the effective action are given by

$$\begin{aligned}
\langle R(z)R(z)\rangle - \langle \mathrm{Tr}(\mathcal{W}'(\Phi)\mathcal{R}(z))\rangle &= 0\,,\\
2\langle R(z)w_\alpha(z)\rangle - \langle \mathrm{Tr}(\mathcal{W}'(\Phi)\rho_\alpha(z))\rangle - \frac{2(32\pi^2)^2\alpha^2}{6}(G\cdot F)_\alpha\langle \mathrm{Tr}(\mathcal{W}'''(\Phi)\mathcal{R}(z))\rangle &= 0\,,\\
2\langle R(z)T(z)\rangle - \langle \mathrm{Tr}(\mathcal{W}'(\Phi)\mathcal{T}(z))\rangle + \langle w^\alpha(z)w_\alpha(z)\rangle + &\\
+\frac{(32\pi^2)^2\alpha^2}{6}\left(-G^2\langle \mathrm{Tr}(\mathcal{W}'''(\Phi)\mathcal{R}(z))\rangle + 2(G\cdot F)^\alpha\langle \mathrm{Tr}\,\mathcal{W}'''(\Phi)\rho_\alpha(z))\rangle\right) &= 0\,.
\end{aligned} \tag{C.4}$$

Here we have used the field redefinition $R(z) \to R(z) + \frac{1}{6}G^2T(z)$ to simplify the equations [8]. Note that the equation for $R(z)$ is unchanged, this is because the terms arising from the variation of S_{Gravity} is already proportional to G^2 or $G\cdot F$, while $\delta\Phi_{ij} = \mathcal{R}_{ij}$. Thus, together they have higher powers of W^2, which can be converted to powers of G using the chiral ring relations in (B.1). As the expansion in powers of G truncates at G^2 and $G\cdot F$, these terms are trivial in the ring.

To obtain the one point functions for $R\,T$ and w_α including the gravitational corrections one needs to solve for the connected correlators. This was done in [8] by deriving equations for the connected correlators using the generalized Konishi anomaly. As the connected correlators are already proportional to either G^2 or $G\cdot F$, the additional terms present in S_{Gravity} do not modify these anomaly equations. Therefore the solutions to the one point functions excluding the $1/\Lambda^4$ corrections are the same as the ones derived in [8], we will note these solutions as $R^{(0)}(z)$, $w^{(0)}_\alpha(z)$ and $T^{(0)}(z)$. They satisfy the following equations

$$\begin{aligned}
\langle R^{(0)}(z)R^{(0)}(z)\rangle - \langle \mathrm{Tr}(\mathcal{W}'(\Phi)\mathcal{R}^{(0)}(z))\rangle &= 0\,,\\
2\langle R^{(0)}(z)w^{(0)}_\alpha(z)\rangle - \langle \mathrm{Tr}(\mathcal{W}'(\Phi)\rho^{(0)}_\alpha(z))\rangle &= 0\,,\\
2\langle R^{(0)}(z)T^{(0)}(z)\rangle - \langle \mathrm{Tr}(\mathcal{W}'(\Phi)\mathcal{T}^{(0)}(z))\rangle + \langle w^{(0)\alpha}(z)w^{(0)}_\alpha(z)\rangle &= 0\,.
\end{aligned} \tag{C.5}$$

where

$$\mathcal{W}' = \sum_{k=1}^{n} g_k\,\mathrm{Tr}(\Phi^k)\,. \tag{C.6}$$

To write down the $1/\Lambda^4$ corrections in a convenient form we will solve the set of equations in (C.5) for $\mathcal{W}' = g_0 + \sum_{k=1}^n g_k\,\mathrm{Tr}(\Phi^k)$ and later set $g_0 = 0$. From the fact the the equation for R does not have any $1/\Lambda^4$ corrections, we see that $R^{(0)}$ continues to be the solution for the one point function in presence of the S_{Gravity}. The corrections to w_α and T satisfy the following equations

$$\begin{aligned}
2\langle R^{(0)}(z)\tilde{w}_\alpha(z)\rangle - \langle \mathrm{Tr}(\mathcal{W}'(\Phi)\tilde{\rho}_\alpha(z))\rangle - \frac{2(32\pi^2)^2\alpha^2}{6}(G\cdot F)_\alpha\langle \mathrm{Tr}(\mathcal{W}'''(\Phi)\tilde{\mathcal{R}}(z))\rangle &= 0\,,\\
2\langle R^{(0)}(z)\tilde{T}(z)\rangle - \langle \mathrm{Tr}(\mathcal{W}'(\Phi)\tilde{\mathcal{T}}(z))\rangle + 2\langle w^{(0)\alpha}(z)\tilde{w}_\alpha(z)\rangle + &\\
+\frac{(32\pi^2)^2\alpha^2}{6}\left(G^2\langle \mathrm{Tr}(\mathcal{W}'''(\Phi)\tilde{\mathcal{R}}(z))\rangle - 2(G\cdot F)^\alpha\langle \mathrm{Tr}\,\mathcal{W}'''(\Phi)\tilde{\rho}_\alpha(z))\rangle\right) &= 0\,.
\end{aligned} \tag{C.7}$$

The solutions of the corrections to the one point function can be constructed from the solutions to $R^{(0)}$ as follows. Let us define the following differential operator

$$\mathcal{D}=\sum_{k=0}^{n}(k+2)(k+1)g_{k+2}\frac{\partial}{\partial g_k}\,. \tag{C.8}$$

$\mathcal{D}$ has the property that $\mathcal{D}\mathcal{W}'(z)=\mathcal{W}'''(z)$. Applying $\mathcal{D}$ on the first and second equation of (C.5) we obtain

$$\begin{aligned}2\langle R^{(0)}(z)\mathcal{D}R^{(0)}(z)\rangle-\langle\mathrm{Tr}(\mathcal{W}'(\Phi)\mathcal{D}\mathcal{R}^{(0)}(z))\rangle-\langle\mathrm{Tr}(\mathcal{W}'''(\Phi)\mathcal{R}^{(0)}(z))\rangle&=0\,,\\ 2\langle R^{(0)}(z)\mathcal{D}w_\alpha^{(0)}(z)\rangle-\langle\mathrm{Tr}(\mathcal{W}'(\Phi)\mathcal{D}\rho_\alpha^{(0)}(z))\rangle+2\langle\mathcal{D}R^{(0)}(z)w_\alpha^{(0)}(z)\rangle-&\\ -\langle\mathrm{Tr}(\mathcal{W}'''(\Phi)\rho_\alpha^{(0)}(z))\rangle&=0\,.\end{aligned} \tag{C.9}$$

Comparing the first equation of (C.7) and (C.9) we see that the corrected solution $\tilde{w}_\alpha(z)$ is given by

$$\tilde{w}_\beta(z)=\alpha^2\frac{2(32\pi)^2}{6}(G\cdot F)_\beta\mathcal{D}R^{(0)}(z)\,. \tag{C.10}$$

Similarly comparing the second equation in (C.7) and the two equations of (C.9) we obtain

$$\tilde{T}(z)=-\frac{\alpha^2(32\pi)^2}{6}G^2\mathcal{D}R^{(0)}(z)+\frac{\alpha^2(32\pi)^2}{6}(G\cdot F)^\alpha\mathcal{D}w_\alpha^{(0)}(z)\,. \tag{C.11}$$

The $1/\Lambda^4$ corrections with $\mathcal{W}'$ given in (C.6) is obtained by evaluating (C.10) and (C.11) and then setting $g_0=0$.

References

[1] R. Dijkgraaf and C. Vafa, *A perturbative window into non-perturbative physics*, hep-th/0208048.

[2] I. Antoniadis, E. Gava, K.S. Narain and T.R. Taylor, *Topological amplitudes in string theory*, *Nucl. Phys.* **B 413** (1994) 162 [hep-th/9307158].

[3] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, *Commun. Math. Phys.* **165** (1994) 311 [hep-th/9309140].

[4] H. Ooguri and C. Vafa, *The C-deformation of gluino and non-planar diagrams*, *Adv. Theor. Math. Phys.* **7** (2003) 53 [hep-th/0302109].

[5] H. Ooguri and C. Vafa, *Gravity induced C-deformation*, *Adv. Theor. Math. Phys.* **7** (2004) 405 [hep-th/0303063].

[6] J. de Boer, P.A. Grassi and P. van Nieuwenhuizen, *Non-commutative superspace from string theory*, *Phys. Lett.* **B 574** (2003) 98 [hep-th/0302078].

[7] N. Seiberg, *Noncommutative superspace, $N=1/2$ supersymmetry, field theory and string theory*, *J. High Energy Phys.* **06** (2003) 010 [hep-th/0305248].

[8] L.F. Alday, M. Cirafici, J.R. David, E. Gava and K.S. Narain, *Gravitational F-terms through anomaly equations and deformed chiral rings*, *J. High Energy Phys.* **01** (2004) 001 [hep-th/0305217].

[9] S. Ferrara, L. Girardello and M. Porrati, *Minimal Higgs branch for the breaking of half of the supersymmetries in $N = 2$ supergravity*, *Phys. Lett.* **B 366** (1996) 155 [hep-th/9510074].

[10] S. Ferrara, L. Girardello and M. Porrati, *Spontaneous breaking of $N = 2$ to $N = 1$ in rigid and local supersymmetric theories*, *Phys. Lett.* **B 376** (1996) 275 [hep-th/9512180].

[11] L. Andrianopoli et al., *$N = 2$ supergravity and $N = 2$ super Yang-Mills theory on general scalar manifolds: symplectic covariance, gaugings and the momentum map*, *J. Geom. Phys.* **23** (1997) 111 [hep-th/9605032].

[12] N. Seiberg, *Adding fundamental matter to 'chiral rings and anomalies in supersymmetric gauge theory'*, *J. High Energy Phys.* **01** (2003) 061 [hep-th/0212225].

[13] P. Fre, L. Girardello, I. Pesando and M. Trigiante, *Spontaneous $N = 2 \to N = 1$ local supersymmetry breaking with surviving compact gauge groups*, *Nucl. Phys.* **B 493** (1997) 231 [hep-th/9607032].

[14] C. Vafa, *Superstrings and topological strings at large-N*, *J. Math. Phys.* **42** (2001) 2798 [hep-th/0008142].

[15] M.R. Douglas and G.W. Moore, *D-branes, quivers and ALE instantons*, hep-th/9603167.

[16] B. de Wit, J.W. van Holten and A. Van Proeyen, *Transformation rules of $N = 2$ supergravity multiplets*, *Nucl. Phys.* **B 167** (1980) 186.

[17] M. de Roo, J.W. van Holten, B. de Wit and A. Van Proeyen, *Chiral superfields in $N = 2$ supergravity*, *Nucl. Phys.* **B 173** (1980) 175.

[18] E. Bergshoeff, M. de Roo and B. de Wit, *Extended conformal supergravity*, *Nucl. Phys.* **B 182** (1981) 173.

[19] E. Cremmer et al., *Vector multiplets coupled to $N = 2$ supergravity: superhiggs effect, flat potentials and geometric structure*, *Nucl. Phys.* **B 250** (1985) 385.

[20] B. de Wit, P.G. Lauwers and A. Van Proeyen, *Lagrangians of $N = 2$ supergravity-matter systems*, *Nucl. Phys.* **B 255** (1985) 569.

[21] I. Antoniadis, H. Partouche and T.R. Taylor, *Spontaneous breaking of $N = 2$ global supersymmetry*, *Phys. Lett.* **B 372** (1996) 83 [hep-th/9512006].

[22] H. Partouche and B. Pioline, *Partial spontaneous breaking of global supersymmetry*, *Nucl. Phys.* **56B** *(Proc. Suppl.)* (1997) 322 [hep-th/9702115].

[23] J. Wess and J. Bagger, *Supersymmetry and supergravity*, Princeton University Press, Princeton 1992.

[24] J.R. David, E. Gava and K.S. Narain, *Konishi anomaly approach to gravitational F-terms*, *J. High Energy Phys.* **09** (2003) 043 [hep-th/0304227].

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